ELEMENTS

OF

ANALYTICAL GEOMETRY

AND OF THE

DIFFERENTIAL AND INTEGRAL

CALCULUS.

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PREFACE.

The following treatise on Analytical Geometry and the Calculus constitutes the fourth volume of a course of Mathematics designed for Colleges and High Schools, and is prepared upon substantially the same model as the preceding volumes. It was written, not for mathematicians, nor for those who have a peculiar talent or fondness for the mathematics, but rather for the mass of college students of average abilities. I have, therefore, labored to smooth down the asperities of the road so as not to discourage travelers of moderate strength and courage; but have purposely left some difficulties, to arouse the energies and strengthen the faculties of the beginner. In a course of liberal education, the primary object in studying the mathematics should be the discipline of the mental powers. This discipline is alike important to the physician and the divine, the jurist and the statesman, and it is more effectually secured by mathematical studies than by any other method hitherto proposed. Hence the mathematics should occupy a prominent place in an education preparatory to either of the learned professions. But, in order to secure the desired advantage, it is indispensable that the student should comprehend the reasons of the processes through which he is conducted. How can he be expected to learn the art of reasoning well, unless he see clearly the foundations of the principles which are taught? This remark applies to every branch of mathematical study, but perhaps to none with the same force as to the Differential and Integral Calculus. The principles of the Calculus are further removed from the elementary conceptions of the mass of mankind than either Algebra, Geometry, or Trigonometry, and they require to be developed with corresponding care. It is quite possible for a student to learn the rules of the Calculus,

and attain considerable dexterity in applying them to the solution of difficult problems, without having acquired any clear idea of the meaning of the terms Differential and Differential Coefficient. Cases of this kind are not of rare occurrence, and the evil may fairly be ascribed, in some degree, to the imperfection of the text-books employed. The English press has for years teemed with "Elementary treatises on the Calculus," many of which are wholly occupied with the mechanical processes of differentiating and integrating, without any attempt to explain the philosophy of these operations. A genuine mathematician may work his way through such a labyrinth, and solve the difficulties which he encounters without foreign assistance; but the majority of students, if they make any progress, will only proceed blindfolded, and after a time will abandon the study in disgust.

I have accordingly given special attention to the development of the fundamental principle of the Differential Calculus, and shall feel a proportionate disappointment if my labors shall be pronounced abortive. The principle from which I have aimed to deduce the whole science, appears to me better adapted to the apprehension of common minds than any other; and although I do not claim for it any originality, it appears to me that I have here developed it in a more elementary manner than I have before seen it presented, except in a small volume by the late Professor Ritchie, of London University. I have derived more important suggestions from this little volume, than from all the other works on the Calculus which have fallen under my notice. The exposition of the principles of the Calculus contained in the following treatise, appears to me so clear, that I indulge the hope that hereafter this subject may be made a standard study for all the students of our colleges, and not be abandoned entirely to the favored few.

While the mental discipline of the majority of students has been the object kept primarily in view, it is believed that the course here pursued will be found best adapted to develop the taste of genuine mathematicians; for a clear conception of the fundamental principles of the science must certainly be favorable to future progress. The student who renders himself familiar with the present treatise will have acquired a degree of

mental discipline which will prove invaluable in every department of business; and he will be enabled, if so inclined, to pursue advantageously any of the standard treatises on the same subject.

Every principle in this work is fully illustrated by practical examples, and at the close will be found a large collection of miscellaneous problems, to be used according to the discretion of the teacher or the taste of the pupil.

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ANALYTICAL GEOMETRY.

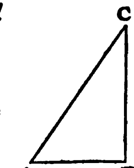
SECTION I.

APPLICATION OF ALGEBRA TO GEOMETRY.

ARTICLE 1. The relations of Geometrical magnitudes may be expressed by means of algebraic symbols, and the demonstrations of Geometrical theorems may thus be exhibited more concisely than is possible in ordinary language. Indeed, so great is the advantage in the use of algebraic symbols, that they are now employed to some extent in all treatises on Geometry.

(2.) The algebraic notation may be employed with even greater advantage in the solution of Geometrical problems. For this purpose we first draw a figure which represents all the parts of the problem, both those which are given and those which are required to be found. The usual symbols or letters for known and unknown quantities are employed to denote both the known and unknown parts of the figure, or as many of them as may be necessary. We then observe the relations which the several parts of the figure bear to each other, from which, by the aid of the proper theorems in Geometry, we derive as many independent equations as there are unknown quantities employed. The solution of these equations by the ordinary rules of algebra will determine the value of the unknown quantities. This method will be illustrated by a few examples.

Ex. 1. *In a right-angled triangle, having given the base and sum of the hypothenuse and perpendicular, to find the perpendicular.*



Let ABC represent the proposed triangle, right-angled at B. Represent the base AB by b, the perpendicular BC by x, and the sum of the hypothenuse and perpendicular by s; then the hypothe-

nuse will be represented by $s-x$. Then, by Geom., Prop. 11, B. IV.,

$$\overline{AB}^2+\overline{BC}^2=\overline{AC}^2;$$

that is,

$$b^2+x^2=(s-x)^2=s^2-2sx+x^2.$$

Taking away x^2 from each side of the equation, we have

$$b^2=s^2-2sx,$$

or

$$2sx=s^2-b^2;$$

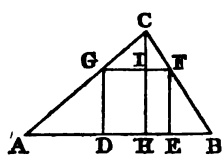
Whence

$$x=\frac{s^2-b^2}{2s},$$

from which we see that in any right-angled triangle, the perpendicular is equal to the square of the sum of the hypothenuse and perpendicular, diminished by the square of the base, and divided by twice the sum of the hypothenuse and perpendicular. Thus, if the base is 3 feet, and the sum of the hypothenuse and perpendicular 9 feet, the expression $\frac{s^2-b^2}{2s}$ becomes $\frac{9^2-3^2}{2\times 9}=4$, the perpendicular.

Ex. 2. *Having given the base and altitude of any triangle, it is required to find the side of the inscribed square.*

Let ABC represent the given triangle, in which there are given the base AB and the altitude CH; it is required to find the side of the inscribed square.



Suppose the inscribed square DEFG to be drawn. Represent the base AB by b, the perpendicular CH by h, and the side of the inscribed square by x; then will CI be represented by $h-x$. Then, because GF is parallel to the base AB, we have, by similar triangles, Geom., Prop. 16, B. IV.,

$$AB : GF :: CH : CI;$$

that is,

$$b : x :: h : h-x;$$

or, since the product of the extremes is equal to that of the means,

$$bh-bx=hx;$$

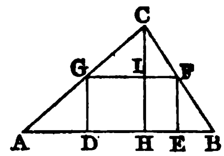
whence

$$x=\frac{bh}{b+h};$$

that is, the side of the inscribed square is equal to the product of the base and height divided by their sum.

Thus, if the base of the triangle is 12 feet, and the altitude 6 feet, the side of the inscribed square is found to be 4 feet.

Ex. 3. *Having given the base and altitude of any triangle, it is required to inscribe within it a rectangle whose sides shall have to each other a given ratio.*



Let ABC be the given triangle, and suppose the required rectangle to be inscribed within it. Represent the base AB by b, the altitude CH by h, the altitude of the rectangle DG by x, and its base DE by y; also, let $x : y :: 1 : n$, or $y=nx$.

Then, because the triangle GFC is similar to the triangle ABC, we have

$$AB : GF :: CH : CI,$$

that is, $$b : y :: h : h-x;$$

whence $$bh-bx=hy.$$

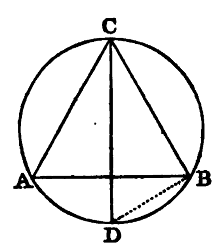
But since $y=nx$, we obtain

$$bh-bx=hnx.$$

Whence $$x=\frac{bh}{b+nh}.$$

If we suppose n equals unity, that is, the sides of the rectangle are equal to each other, the preceding result becomes identical with that in Example 2.

Ex. 4. *The diameter of a circle being given, to determine the side of the inscribed equilateral triangle.*



Suppose ABC to be the required triangle inscribed in a circle whose diameter is CD. Represent CD by d, and CB by x. Also join DB. Then, Geom., Prop. 15, Cor. 2, B. III., CBD is a right-angled triangle, and, Geom., Prop. 4, B. VI., BD is one half of CD.

Hence $$CB^2+BD^2=CD^2;$$

that is, $$x^2+\frac{d^2}{4}=d^2.$$

Whence $$x^2=\frac{3d^2}{4},$$

or $$x=\frac{d\sqrt{3}}{2};$$

that is, the side of the inscribed triangle is equal to the diameter of the circle multiplied by half the square root of three.

Ex. 5. The diameter, d, of a circle being given, to determine the side of the circumscribed equilateral triangle.

Ans. $d\sqrt{3}$.

Ex. 6. From any point within an equilateral triangle, perpendiculars are drawn to the three sides. It is required to find the sum, s, of these perpendiculars.

Ans. $s=$ altitude of the triangle.

Ex. 7. Given the difference, d, between the diagonal of a square and one of its sides, to find the length of the sides.

Ans. $d+d\sqrt{2}$.

Ex. 8. In a right-angled triangle, the lines a and b, drawn from the acute angles to the middle of the opposite sides, are given, to find the lengths of the sides.

Ans. $2\sqrt{\dfrac{4b^2-a^2}{15}}$, and $2\sqrt{\dfrac{4a^2-b^2}{15}}$.

Ex. 9. Given the lengths of three perpendiculars, a, b, and c, drawn from a certain point in an equilateral triangle to the three sides, to find the length of the sides.

Ans. $\dfrac{2(a+b+c)}{\sqrt{3}}$.

Ex. 10. Given the base b and the difference d between the hypothenuse and perpendicular of a right-angled triangle, to find the perpendicular.

Ans. $\dfrac{b^2-d^2}{2d}$.

Ex. 11. Given the hypothenuse h of a right-angled triangle, and the ratio of the base to the perpendicular, as m to n, to find the perpendicular.

Ans. $\dfrac{nh}{\sqrt{m^2+n^2}}$.

Ex. 12. Given the diagonal d of a rectangle, and the perimeter $4p$, to find the lengths of the sides.

Ans. $p\pm\sqrt{\dfrac{d^2}{2}-p^2}$.

Ex. 13. If the diagonal of a rectangle be 10 feet, and its pe rimeter 28 feet, what are the lengths of the sides?

Ans.

SECTION II.

CONSTRUCTION OF EQUATIONS.

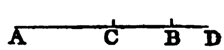
(3.) The construction of an equation consists in finding a Geometrical figure which may be considered as representing that equation; that is, a figure in which the relation between the parts shall be the same as that expressed by the equation.

Problem I. To construct the equation $x=a+b$.

The symbols a and b being supposed to stand for numerical quantities may be represented by lines. The length of a line is determined by comparing it with some known standard, as an inch or a foot. If the line AB contains the standard unit a times, then AB may be taken to represent a. So, also, if BC contains the standard unit b times, then BC may be taken to represent b. Therefore, in order to construct the expression $a+b$, draw an indefinite line AD. From the point A lay off a distance AB equal to a, and from B lay off a distance BC equal to b, then AC will be a right line representing $a+b$.

Problem II. To construct the equation $x=a-b$.

Draw the indefinite line AD. From the point A lay off a distance AB equal to a, and from B lay off a distance BC in the direction toward A equal to b, then will AC be the difference between AB and BC; consequently, it may be taken to represent the expression $a-b$.

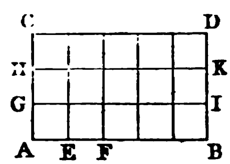


(4.) A single factor may always be represented by a *line*, and an algebraic expression, consisting of a series of letters connected together by the signs + and −, may be represented by drawing a line of indefinite length, and setting off upon it all the positive terms in one direction, and all the negative terms in the opposite direction.

Problem III. To construct the equation $x=ab$.

Let ABCD be a rectangle of which the side AB contains the standard unit a times, and the side AC contains the same unit

b times. If we draw lines parallel to AC through the points E, F, etc., and lines parallel to AB through the points G, H, etc., the rectangle will be divided into square units. Then, in the first row, AGIB, there are a square units; in the second row, GHKI, there are also a square units, and there are as many rows as there are units in AC; therefore the rectangle contains $a\times b$ square units, or the rectangle may be considered as representing the expression ab.



(5.) The product of two factors may, therefore, always be represented by a *surface*.

PROBLEM IV. To construct the equation $x=abc$.

Let there be a parallelopiped whose three adjacent edges contain the standard unit respectively a, b, and c times; then, dividing the solid by planes parallel to its sides, we may prove that the number of solid units in the figure is $a\times b\times c$, and, consequently, the parallelopiped may be considered as representing the expression abc.

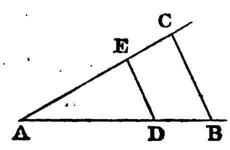
(6.) The product of three factors may, therefore, always be represented by a *solid*.

PROBLEM V. To construct the equation $x=\frac{ab}{c}$.

If $x=\frac{ab}{c}$, then $\quad c : a :: b : x;$

that is, x is a fourth proportional to the three given quantities, c, a, and b; hence the line whose length is expressed by x is a fourth proportional to three lines whose respective lengths are c, a, and b.

From A draw two lines AB, AC making any angle with each other. From A lay off a distance AD equal to c, AB equal to a, and AE equal to b. Join DE, and through B draw BC parallel to DE; then will AC be equal to x.



For, by similar triangles, we have

$$AD : AB :: AE : AC,$$

or $$c : a :: b : AC.$$

Hence $$AC=\frac{ab}{c}.$$

PROBLEM VI. To construct the equation $x=\frac{abc}{de}$.

This expression can be put under the form

$$\frac{ab\times c}{d\times e}; \text{ or } \frac{ab}{d}\times\frac{c}{e}.$$

First find a fourth proportional m to the three quantities d, a, and b; that is, make

$$d : a :: b : m; \text{ whence } m=\frac{ab}{d}.$$

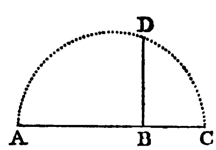
The proposed expression then becomes

$$\frac{mc}{e},$$

which may be constructed as in Problem V.

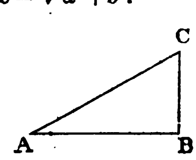
PROBLEM VII. To construct the equation $x=\sqrt{ab}$.

Since $\sqrt{ab}$ is a mean proportional between a and b, the problem requires us to find geometrically a mean proportional between a and b.



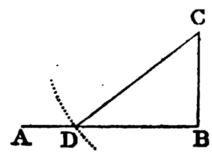
Draw an indefinite straight line, and upon it set off AB equal to a, and BC equal to b. On AC, as a diameter, describe a semicircle, and from B draw BD perpendicular to AC, meeting the circumference in D; then BD is a mean proportional between AB and BC (Geom., Prop. 22, Cor., B. IV.). Hence BD is a line representing the expression $\sqrt{ab}$.

PROBLEM VIII. To construct the equation $x=\sqrt{a^2+b^2}$.



Draw the line AB, and make it equal to a; from B draw BC perpendicular to AB, and make it equal to b. Join AC, and it will represent the value of $\sqrt{a^2+b^2}$. For $AC^2=AB^2+BC^2$ (Geom., Prop. 11, B. IV.).

PROBLEM IX. To construct the equation $x=\sqrt{a^2-b^2}$.



Draw an indefinite right line AB; at B draw BC perpendicular to AB, and make it equal to b. With C as a center, and a radius equal to a, describe an arc of a circle cutting AB in D; then will BD represent the expression $\sqrt{a^2-b^2}$. For

$$BD^2=DC^2-BC^2=a^2-b^2;$$

Whence $$BD=\sqrt{a^2-b^2}.$$

SECTION II.

SUCCESSIVE DIFFERENTIALS—MACLAURIN'S THEOREM—TAYLOR'S THEOREM—FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES.

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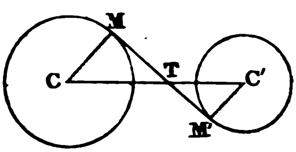
INTEGRATION OF MONOMIAL DIFFERENTIALS—OF BINOMIAL DIFFERENTIALS—OF THE DIFFERENTIALS OF CIRCULAR ARCS.

constant, and the smaller radius r' to increase, the difference $r-r'$ will diminish; and, since the numerator ar remains constant, the value of x will increase; which shows that the nearer the two circles approach to equality, the more distant is the point of intersection of the tangent line with the line joining the centers. When the two radii r and r' become equal, the denominator becomes 0, and the value of x becomes infinite.

Cor. 2. If we suppose r' to increase so as to become greater than r, the value of x becomes negative, which shows that the point T falls to the left of the two circles.

Cor. 3. Two other tangent lines may be drawn intersecting each other between the circles. If we represent CT by x, the radii of the circles by r and r', and the distance between their centers by a, we shall have from the similar triangles CMT, C′M′T, the proportion

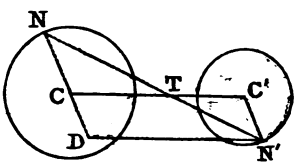


$$\text{CM} : \text{C}'\text{M}' :: \text{CT} : \text{C}'\text{T},$$

or
$$r : r' :: x : a-x;$$

whence
$$x=\frac{ar}{r+r'}.$$

This expression may be constructed in a manner similar to the former. Through the centers C and C′ draw two parallel radii CN, C′N′, lying on different sides of the line CC′; join the points NN′, and through T, where this line intersects CC′, draw a line tangent to one of the circles, it will be a tangent to the other. For through N′ draw N′D parallel to CC′, and meeting CN produced in D. DN′ will then represent a, ND will represent $r+r'$, and the similar triangles NCT, NDN′ will furnish the proportion



$$\text{ND} : \text{DN}' :: \text{NC} : \text{CT},$$

or
$$r+r' : a :: r : \text{CT};$$

whence
$$\text{CT}=\frac{ar}{r+r'};$$

which is the value already found for x.

(7.) Every Algebraic expression, admitting of geometrical construction, must have all its terms *homogeneous* (Algebra,

Art. 31); that is, each term must contain the same number of literal factors. Thus, each term must either be of one dimension, and so represent a line; or, secondly, each must be of two dimensions, and represent a surface; or, thirdly, each must be of three dimensions, and denote a solid; since dissimilar geometrical magnitudes can neither be added together nor subtracted from each other.

It may, however, happen that an expression really admitting of geometrical construction appears to be not homogeneous; but this result arises from the circumstance that the geometrical unit of length, being represented algebraically by 1, disappears from all algebraic expressions in which it is either a factor or a divisor. To render these results homogeneous, it is only necessary to restore this divisor or factor which represents unity.

Thus, suppose we have an equation of the form

$$x=ab+c.$$

If we put l to represent the unit of measure for lines, we may change it into the homogeneous equation,

$$lx=ab+cl,$$

or

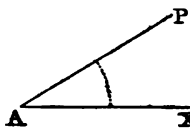
$$x=\frac{ab}{l}+c;$$

which is easily constructed geometrically.

SECTION III.

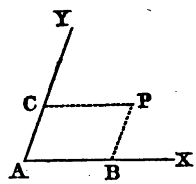
ON THE POINT AND STRAIGHT LINE.

(8.) There are two methods of denoting the position of a point in a plane. The first is by means of the distance and direction of the proposed point from a given point. Thus, if A be a known point, and AX be a known direction, the position of the point P will be determined when we know the distance AP and the angle PAX.



The assumed point A is called the *pole;* the distance of P from A is called the *radius vector;* and the radius vector, together with its angle of inclination to the fixed line, are called the *polar co-ordinates* of the point.

(9.) It is, however, generally most convenient to denote the position of a point by means of its distances from two given lines which intersect one another. Thus, let AX, AY be two assumed straight lines which intersect in any angle at A, and let P be a point in the same plane; then, if we draw PB parallel to AY, and PC parallel to AX, the position of the point P will be denoted by means of the distances PB and PC.



The two lines AX, AY, to which the position of the point P is referred, are called *axes,* and their point of intersection A is called their *origin.* The distance AB, or its equal CP, is called the *abscissa* of the point P; and BP, or its equal AC, is called the *ordinate* of the same point. Hence the axis AX is called the *axis of abscissas,* and AY is called the *axis of ordinates.*

The abscissa and ordinate of a point, when spoken of together, are called the *co-ordinates* of the point, and the two axes are called *co-ordinate axes.*

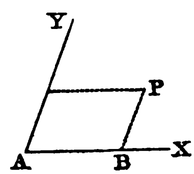
The axes are called oblique or rectangular, according as YAX is an oblique or a right angle. Rectangular axes are the most simple, and will generally be employed in this treatise.

An abscissa is generally denoted by the letter x, and an ordinate by the letter y; and hence the axis of abscissas is often called the axis of X, and the axis of ordinates the axis of Y.

The abscissa of any point is its distance from the axis of ordinates, measured on a line parallel to the axis of abscissas.

The ordinate of any point is its distance from the axis of abscissas, measured on a line parallel to the axis of ordinates.

(10.) The position of a point may be determined when its co-ordinates are known. For suppose the abscissa of the point P is equal to a, and its ordinate is equal to b. Then, to determine the position of the point P, from the origin A lay off on the axis of abscissas a distance AB equal to a, and through B draw a line parallel to the axis of ordinates. On this line lay off a distance BP equal to b, and P will be the point required.

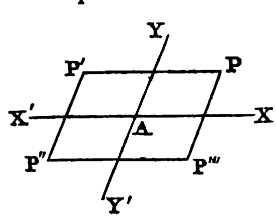


Hence, in order to determine the position of a point, we need only have the two equations

$$x=a,\ y=b,$$

in which a and b are given. These equations are, therefore, called *the equations of a point.*

(11.) It is, however, necessary, in order to determine the position of a point, that not only the absolute values of a and b should be given, but also the signs of these quantities. If the axes are produced through the origin to X′ and Y′, it is obvious that the abscissas reckoned in the direction AX′ ought not to have the same sign as those reckoned in the opposite direction AX; nor should the ordinates measured in the direction AY′ have the same sign as those measured in the opposite direction AY; for if there were no distinction in this respect, the position of a point as determined by its equations would be ambiguous. Thus the equations of the point P would equally belong to the points P′, P″, P‴, provided the absolute lengths of the co-ordinates of each were equal to those of P. All this ambiguity is avoided by regarding the co-ordinates which are measured in one direction as *plus,* and those in the opposite direction *minus.* It is generally agreed to regard those abscissas which



fall on the right of the origin A as positive, and hence those which fall on the left must be considered negative. So, also, it has been agreed to consider those ordinates which are above the origin as positive, and hence those which fall below it must be considered negative.

(12.) The angle YAX is called the *first angle;* YAX′ the *second angle;* Y′AX′ the *third angle;* and Y′AX the *fourth angle.*

The following, therefore, are the equations of a point in each of the four angles:

For the point	P	in the first angle,	$x=+a, y=+b$,
"	P′	" second angle,	$x=-a, y=+b$,
"	P″	" third angle,	$x=-a, y=-b$,
"	P‴	" fourth angle,	$x=+a, y=-b$.

If the point be situated on the axis AX, the equation $y=b$ becomes $y=0$, so that the equations

$$x=\pm a, y=0,$$

characterize a point on the axis of abscissas at the distance a from the origin.

If the point be situated on the axis AY, the equation $x=a$ becomes $x=0$, so that the equations

$$x=0, y=\pm b,$$

characterize a point on the axis of ordinates at the distance b from the origin.

If the point be common to both axes, that is, if it be at the origin, its position will be expressed by the equations

$$x=0, y=0.$$

Ex. 1. Determine the point whose equations are $x=+4$, $y=-3$.

Ex. 2. Determine the point whose equations are $x=-2$, $y=+7$.

Ex. 3. Determine the point whose equations are $x=0$, $y=-5$.

Ex. 4. Determine the point whose equations are $x=-8$, $y=0$.

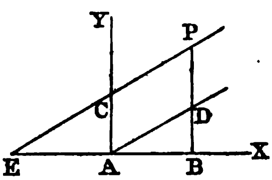
DEFINITION.—*The equation of a line is the equation which expresses the relation between the co-ordinates of every point of the line.*

PROPOSITION I.—THEOREM.

(13.) *The equation of a straight line referred to rectangular axes is*

$$y=ax+b;$$

where x and y are the co-ordinates of any point of the line, a represents the tangent of the angle which the line makes with the axis of abscissas, and b the distance from the origin at which it intersects the axis of ordinates. Also, a and b may be either positive or negative.



Let A be the origin of co-ordinates, AX and AY be rectangular axes, and PC any straight line whose equation is required to be determined. Take any point P in the given line, and draw PB perpendicular to AX; then will PB be the ordinate, and AB the abscissa of the point P. From A draw AD parallel to CP, meeting the line BP in D.

Let $\quad\quad$ $AB=x$,

$BP=y$,

tangent PEX or $DAX=a$,

and $\quad\quad$ AC or $DP=b$.

Then, by Trigonometry, Theorem II., Art. 42,

$$R : AB :: \text{tang. } DAX : BD,$$

or

$$R : x :: a : BD.$$

Hence, calling the radius unity, we have

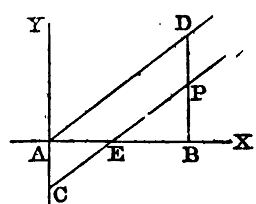
$$BD=ax.$$

But

$$BP=BD+DP;$$

hence

$$y=ax+b.$$

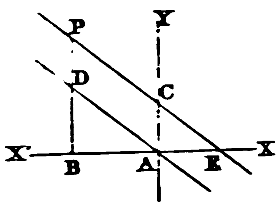


If the line CP cuts the axis of ordinates below the origin, then we shall have $\quad BP=BD-DP$,

or $\quad y=ax-b$.

(14.) The line PC has been drawn so as to make an acute angle with the axis of abscissas; but the preceding equation is equally applicable whatever may be this angle, provided proper signs are attributed to each term. The angle which the line makes with the axis of abscissas is supposed to be measured from the axis AX around the circle by the left.

If the angle is obtuse, its tangent will be negative (Trigonometry, Art. 70). Thus, if PC be the position of the proposed line with reference to the rectangular axes AX, AY, then, in the proportion



$$R : AB :: \text{tang. } DAX : BD,$$

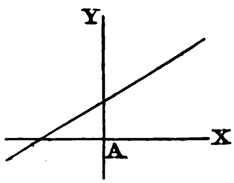
the abscissa AB is negative, being measured from the origin toward the left; the tangent of DAX is also negative; their product is therefore positive, as it should be, for the ordinate BD is positive, being measured from the origin above the axis of abscissas. The equation of this line may, therefore, be written

$$y=-ax+b,$$

where it must be observed that the sign $-$ applies only to the quantity a, and not to x, for the sign of x depends upon its direction from the origin A. If the line PC be produced toward the right beyond AY, its abscissas will be positive.

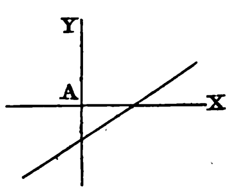
(15.) There may, therefore, be four positions of the proposed line, and these positions are indicated by the signs of a and b in the general equation.

1. Let the line take the position shown in the annexed diagram, cutting the axis of X to the left of the origin, and the axis of Y above it, then a and b are both positive, and the equation is



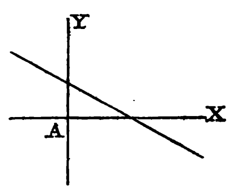
$$y=+ax+b.$$

2. If the line cuts the axis of X to the right of the origin, and the axis of Y below it, then a will still be positive, but b will be negative, and the equation becomes



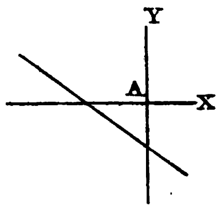
$$y=+ax-b.$$

3. If the line cuts the axis of X to the right of the origin, and the axis of Y above it, then a becomes negative and b positive. In this case, therefore, the equation is



$$y=-ax+b.$$

4. If the line cuts the axis of X to the left of the origin, and

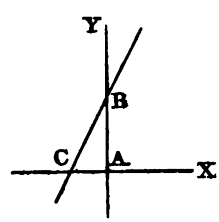


the axis of Y below it, then both a and b will be negative, so that the equation becomes $$y=-ax-b.$$

If we suppose the straight line to pass through the origin A, then b will be equal to zero, and the general equation becomes
$$y=ax,$$
which is the equation of a straight line passing through the origin.

Ex. 1. Let it be required to draw the line whose equation is
$$y=2x+4.$$

If in this equation we make $x=0$, the value of y will designate the point in which the line intersects the axis of ordinates, for that is the only point of the line whose abscissa is 0. This supposition will give
$$y=4.$$



Having drawn the co-ordinate axes AX, AY, lay off from the origin A a distance AB equal to 4; this will be one point of the required line.

Again, if in the proposed equation we make $y=0$, the value of x, which is found from the equation, will designate the point in which the line intersects the axis of abscissas, for that is the only point of the line whose ordinate is 0. This supposition will give $$2x=-4,$$
or $$x=-2.$$

Lay off from the origin A, toward the left, a distance AC equal to 2; this will give a second point of the proposed line, and the line may be drawn through the two points B and C.

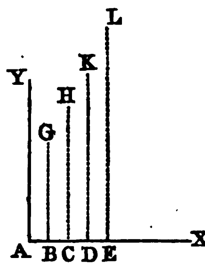
(16.) We may determine any number of points in this line by assuming particular values for x or y; the equation will furnish the corresponding value of the other variable.

Making successively

$x=1$, we find $y=6$,
$x=2$, " $y=8$,
$x=3$, " $y=10$,
$x=4$, " $y=12$,
etc. etc.

In order to represent these values by a figure, we draw two

axes AX, AY at right angles to each other. Then, in order to construct the values $x=1$, $y=6$, we set off on the axis of abscissas a line AB equal to 1, and erect a perpendicular BG equal to 6; this determines one point of the required line. Again, take AC equal to 2, and make the perpendicular CH equal to 8; this will determine a second point of the required line. In the same manner we may determine the points K and L, and any number of points. The required line must pass through all the points, G, H, K, L, etc.



Ex. 2. Construct the line whose equation is

$$y=2x+3.$$

Ex. 3. Construct the line whose equation is

$$y=3x-7.$$

Ex. 4. Construct the line whose equation is

$$y=-x+2.$$

Ex. 5. Construct the line whose equation is

$$y=-2x-5.$$

In the equation $y=ax+b$, the quantities a and b remain the same, while the co-ordinates x and y vary in value for every point in the same line. We, therefore, call a and b *constant* quantities, and x and y *variable* quantities.

Proposition II.—Theorem.

(17.) *Every equation of the first degree containing two variables is the equation of a straight line.*

Every equation of the first degree containing two variables can be reduced to the form

$$Ay=Bx+C,$$

whence

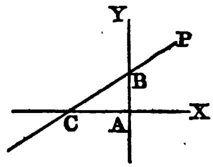
$$y=\frac{B}{A}x+\frac{C}{A}.$$

For the sake of simplicity, let us put a for $\frac{B}{A}$, and b for $\frac{C}{A}$, and the equation reduces to

$$y=ax+b. \qquad (1)$$

Draw the co-ordinate axes AX, AY at right angles to each

other; make AB equal to b, and AC equal to $\frac{b}{a}$, and through the points B and C draw the line PBC, it will be the geometrical representation of the proposed equation.



For the equation of this line is

$$y=\frac{\text{AB}}{\text{AC}}x+\text{AB}; \qquad (2)$$

but by the construction,

$$\frac{\text{AB}}{\text{AC}}=b\div\frac{b}{a}=a;$$

also, $\text{AB}=b$.

Therefore the equations (1) and (2) are identical, and each is the analytical representation of the line PBC.

Ex. Draw the line whose equation is $2y=3x-5$.

PROPOSITION III.—THEOREM.

(18.) *The equation of a straight line passing through a given point is*

$$y-y'=a(x-x'),$$

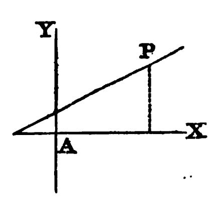
where x' and y' denote the co-ordinates of the given point, x and y the co-ordinates of any point of the line, and a the tangent of the angle which the line makes with the axis of abscissas.

Known co-ordinates are usually designated by marking them thus,

$$x', y'; \ x'', y''; \ x''', y''', \text{ etc.},$$

which are read x prime, y prime; x second, y second; x third, y third, etc.

Let P be the given point, and designate its co-ordinates by x' and y'. Then, since the general equation for every point in the required line is



$$y=ax+b, \qquad (1)$$

it follows that when the variable abscissa x becomes x', then y will become y'; hence the equation to the line becomes

$$y'=ax'+b. \qquad (2)$$

Equation (1) may be regarded as containing four unknown

quantities, and equation (2) imposes a new condition upon them. Each of these equations must be satisfied at the same time, and by combining them in one, we shall be able to eliminate one of the unknown quantities.

Subtracting equation (2) from equation (1), we obtain

$$y-y'=a(x-x'),$$

which is the equation of a line passing through the given point P.

Since the tangent a, which fixes the direction of the line, is not determined, there may be an infinite number of straight lines drawn through a given point. This is also apparent from the figure.

(19.) If it be required that the line shall pass through a given point, and be parallel to a given line, then the angle which the line makes with the axis of abscissas is determined; and if we put a' for the tangent of this angle, the equation of the line sought will be

$$y-y'=a'(x-x').$$

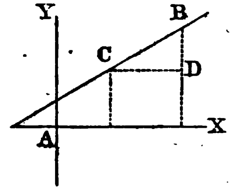
Ex. Draw a line through the point whose abscissa is 5 and ordinate 3, making an angle with the axis of abscissas whose tangent is equal to 2.

Proposition IV.—Theorem.

(20.) *The equation of a straight line which passes through two given points is*

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'),$$

where x' and y' are the co-ordinates of one of the given points, x'' and y'' the co-ordinates of the other point, and x and y the general co-ordinates of the line.



Let B and C be the two given points, the co-ordinates of B being x' and y', and the co-ordinates of C being x'' and y''. Then, since the general equation for every point in the required line is

$$y=ax+b, \quad (1)$$

it follows that when the variable abscissa x becomes x', then y will become y'; hence the equation to the line becomes

$$y'=ax'+b. \quad (2)$$

Also, when the variable abscissa x becomes x'', then y becomes y'', and the equation to the line becomes

$$y''=ax''+b. \qquad (3)$$

Equation (1) contains four unknown quantities, and equations (2) and (3) impose two new conditions upon them. By combining these three equations in one, we shall be able to eliminate two of the unknown quantities; and this we are at liberty to do, since each of these equations must be satisfied at the same time.

If we subtract equation (2) from equation (1), we obtain

$$y-y'=a(x-x'). \qquad (4)$$

Also, if we subtract equation (3) from equation (2), we obtain

$$y'-y''=a(x'-x''),$$

from which we find $$a=\frac{y'-y''}{x'-x''}.$$

Substituting this value of a in equation (4), we have

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'),$$

which is the equation of the line passing through the two given points B and C.

(21.) We have found a equal to $\frac{y'-y''}{x'-x''}$. This is obvious from the figure. For $y'-y''$ is equal to BD, and $x'-x''$ is equal to CD; hence $\frac{y'-y''}{x'-x''}$ is equal to $\frac{\mathrm{BD}}{\mathrm{CD}}$, which is the tangent of the angle BCD, the radius being unity (Trigonometry, Art. 42).

If the proposed line passes through the origin, then $x''=0$, and $y''=0$, and the equation becomes

$$y=\frac{y'}{x'}x,$$

which is the equation of a straight line passing through the origin and through a given point.

Ex. 1. Find the equation to the straight line which passes through the two points whose co-ordinates are $x'=7$, $y'=4$, $x''=5$, $y''=3$, and determine the angle which it makes with the axis of abscissas.

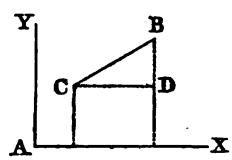
Ex. 2. Find the equation to the straight line which passes through the two points $x'=2$, $y'=3$, and $x''=4$, $y''=5$.

PROPOSITION V.—THEOREM.

(22.) *The distance between two given points is equal to*

$$\sqrt{(x'-x'')^2+(y'-y'')^2},$$

where x' and y' are the co-ordinates of one of the given points, and x'' and y'' those of the other.



Let B and C be the two given points. Designate the co-ordinates of B by x' and y', and the co-ordinates of C by x'' and y''. Draw CD parallel to AX. The distance BC is equal to

$$\sqrt{CD^2+BD^2}.$$

But $CD=x'-x''$, and $BD=y'-y''$; therefore the expression for the distance between B and C is

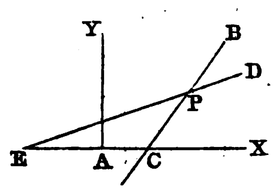
$$\sqrt{(x'-x'')^2+(y'-y'')^2}.$$

PROPOSITION VI.—THEOREM.

(23.) *The tangent of the angle included between two straight lines is*

$$\frac{a'-a}{1+aa'},$$

where a and a' denote the tangents of the angles which the two lines make with the axis of abscissas.



Let BC and DE be any two lines intersecting each other in P. Let the equation of the line DE be

$$y=ax+b,$$

and the equation of BC be

$$y=a'x+b';$$

then a will be the tangent of angle PEX, and a' the tangent of the angle PCX. Designate the angle PEX by α, and the angle PCX by α'. Now, because PCX is the exterior angle of the triangle PEC, it is equal to the sum of the angles CPE and PEC; that is, the angle EPC is equal to the difference of the angles PCX and PEX, or

$$EPC=PCX-PEX=\alpha'-\alpha;$$

whence tang. EPC=tang. (PCX−PEX)=tang. $(\alpha'-\alpha)$.

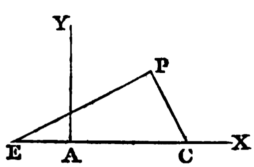
But, by Trigonometry, Art. 76,

$$\text{tang. }(\alpha'-\alpha)=\frac{\text{tang. }\alpha'-\text{tang. }\alpha}{1+\text{tang. }\alpha\text{ tang. }\alpha'}.$$

Therefore, $$\text{tang. EPC}=\frac{a'-a}{1+aa'}.$$

If the angle of intersection of the two lines be a right angle, its tangent must be infinite. But in order that the expression $\frac{a'-a}{1+aa'}$ may become infinite, the denominator $1+aa'$ must become zero; so that in this case we must have $aa'=-1$, or $a=-\frac{1}{a'}$. This, then, is the condition by which two straight lines are shown to be at right angles to each other.

(24.) This last conclusion might have been derived from the principles of Trigonometry. Thus, let the two lines PC, PE be perpendicular to each other; then the angle PCE is the complement of PEC. But by (Trig., Art. 28) tang.$\times$cotang.$=\text{R}^2$ or unity; hence tang. PEC$\times$tang. PCE $=1$. Now PCX, being the supplement of PCE, has the same tangent (Trig., Art. 27), but with a negative sign (Trig., Art. 70). Hence



$$\text{tang. PEC}\times\text{tang. PCX}=-1.$$

(25.) We have found, Art. 18, that the equation of a straight line passing through a given point is

$$y-y'=a(x-x').$$

To find the equation of a line perpendicular to it, we must substitute for a, $-\frac{1}{a'}$. Hence

$$y-y'=-\frac{1}{a'}(x-x'),$$

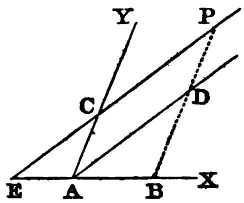
is the equation of a line passing through a given point, and perpendicular to a given line.

PROPOSITION VII.—THEOREM.

(26.) *The equation of a straight line referred to oblique axes is*

$$y=ax+b,$$

where a represents the ratio of the sine of the angle which the line makes with the axis of abscissas, to the sine of the angle which it makes with the axis of ordinates.



Let A be the origin of co-ordinates, and AX, AY oblique axes, and PC any straight line whose equation is required to be determined. Take any point P in the given line, and draw PB parallel to AY; then will PB be the ordinate, and AB the abscissa of the point P. From A draw AD parallel to CP, meeting the line BP in D. Denote the angle PEX, or its equal DAX, by α, and the angle YAX by β.

Since PB is parallel to AY, the angle ADB is equal to DAY; that is, equal to $\beta-\alpha$.

Let $$AB=x,$$
$$BP=y,$$
and $$AC \text{ or } DP=b.$$

Then, by Trigonometry, Theorem I., Art. 49,

$$BD : AB :: \sin. \alpha : \sin. (\beta-\alpha),$$

or $$BD : x :: \sin. \alpha : \sin. (\beta-\alpha).$$

Hence $$BD=x\frac{\sin. \alpha}{\sin. (\beta-\alpha)}.$$

But $$BP=BD+DP.$$

Hence $$y=x\frac{\sin. \alpha}{\sin. (\beta-\alpha)}+b.$$

The coefficient of x in this equation is equal to the sine of the angle which the line makes with the axis of X, divided by the sine of the angle which it makes with the axis of Y; and if we represent this factor by a, the equation may be written

$$y=ax+b,$$

which is of the same form as in Theorem I., but the factor a has a different signification.

ON THE TRANSFORMATION OF CO-ORDINATES.

(27.) When a line is represented by an equation in reference to any system of axes, we can always transform that equation into another which shall equally represent the line, but in reference to a new system of axes chosen at pleasure. This is

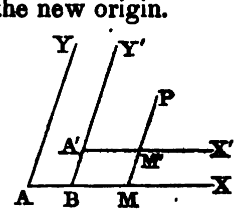
called the transformation of co-ordinates; and may consist either in altering the relative position of the axes without changing the origin, or changing the origin without disturbing the relative position of the axes; or we may change both the direction of the axes and the position of the origin.

PROPOSITION VIII.—THEOREM.

(28.) *The formulas for passing from one system of co-ordinate axes to another system, respectively parallel to the first, are,*

$$x=a+x',$$
$$y=b+y',$$

in which a and b are the co-ordinates of the new origin.



Let AX, AY be the primitive axes, and let A′X′, A′Y′ be the new axes to which it is proposed to refer the same line.

Let the co-ordinates of the new origin AB, A′B be represented by a and b; let the co-ordinates of any point P relative to the primitive axes be x and y, and the co-ordinates of the same point referred to the new axes be x' and y'. Then we shall have

$$\text{AM}=\text{AB}+\text{BM}, \text{ and } \text{PM}=\text{MM}'+\text{PM}';$$

that is, $x=a+x'$, and $y=b+y'$,

which are the equations required.

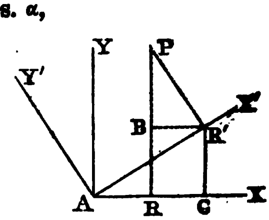
The new origin A′ may be placed in either of the four angles of the primitive system, by attributing proper signs to a and b.

PROPOSITION IX.—THEOREM.

(29.) *The formulas for passing from a system of rectangular co-ordinates to another system also rectangular are,*

$$x=x' \cos. \alpha-y' \sin. \alpha,$$
$$y=x' \sin. \alpha+y' \cos. \alpha,$$

where α represents the angle included between the two axes of X.



Let AX, AY be the primitive axes, and AX′, AY′ be the new axes, and let us designate the co-ordinates of the point P referred to the primitive axes

by x and y, and its co-ordinates referred to the new axes by x', y'. Denote the angle XAX′ by α. Through P draw PR perpendicular to AX, and PR′ perpendicular to AX′; draw R′C perpendicular, and R′B parallel to AX.

Then $\quad AR=AC-CR.$
But $\quad AR=x.$
Also, $\quad AC=AR' \cos.\times XAX'=x' \cos. \alpha,$
and $\quad CR=BR'=PR' \sin. BPR'=y' \sin. \alpha.$
Hence $\quad x=x' \cos. \alpha-y' \sin. \alpha.$
Also, $\quad PR=BR+PB.$
But $\quad PR=y;$
$\quad BR=R'C=AR' \sin. XAX'=x' \sin. \alpha;$
and $\quad PB=PR' \cos. BPR'=y' \cos. \alpha.$
Hence $\quad y=x' \sin. \alpha+y' \cos. \alpha.$

SCHOLIUM. If the origin be changed at the same time to a point whose co-ordinates, when referred to the primitive system, are a and b, these equations will become

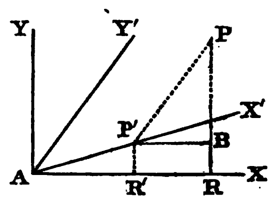
$$x=a+x' \cos. \alpha-y' \sin. \alpha,$$
$$y=b+x' \sin. \alpha+y' \cos. \alpha.$$

PROPOSITION X.—THEOREM.

(30.) *The formulas for passing from a system of rectangular, to a system of oblique co-ordinates, are,*

$$x=x' \cos. \alpha+y' \cos. \alpha',$$
$$y=x' \sin. \alpha+y' \sin. \alpha',$$

where α and α' denote the inclination of the new axes to the primitive axis of abscissas.



Let AX, AY be the primitive axes, AX′, AY′ the new axes. Denote the angle XAX′ by α, and the angle XAY′ by α'. Through P draw PR parallel to AY, and PP′ parallel to AY′; draw, also, P′R′ parallel to AY, and P′B parallel to AX.

Then $\quad AR=AR'+R'R.$
But $\quad AR=x,$
$\quad AR'=AP' \cos. XAX'=x' \cos. \alpha,$
and $\quad R'R=P'B=PP' \cos. BP'P=y' \cos. \alpha'.$
Hence $\quad x=x' \cos. \alpha+y' \cos. \alpha'.$

Also, $PR = BR + PB$.

But $PR = y$,

$BR = P'R' = AP' \sin. XAX' = x' \sin. \alpha$,

and $PB = PP' \sin. PP'B = y' \sin. \alpha'$.

Hence $y = x' \sin. \alpha + y' \sin. \alpha'$.

SCHOLIUM. If the origin be changed at the same time to a point whose co-ordinates, referred to the primitive system, are a and b, these equations will become

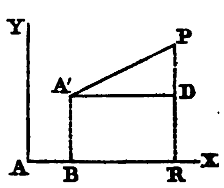
$$x = a + x' \cos. \alpha + y' \cos. \alpha',$$
$$y = b + x' \sin. \alpha + y' \sin. \alpha'.$$

PROPOSITION XI.—THEOREM.

(31.) *The formulas for passing from a system of rectangular to a system of polar co-ordinates are,*

$$x = a + r \cos. v,$$
$$y = b + r \sin. v,$$

where r denotes the radius vector, and v the angle which it makes with the axis of abscissas.



Let AX, AY be the primitive axes, A′ the pole, and A′D, parallel to AX, be the line from which the variable angle is to be estimated.

Designate the angle PA′D by v, the radius vector A′P by r, the co-ordinates of the point P referred to the primitive axes by x and y, and the co-ordinates of A′ by a and b.

Now $AR = AB + BR$.

But $BR = A'D = A'P \cos. PA'D = r \cos. v$.

Hence $x = a + r \cos. v$.

Also, $PR = DR + PD$.

But $PD = A'P \sin. PA'D = r \sin. v$.

Hence $y = b + r \sin. v$.

SCHOLIUM. If the pole A′ be placed at the origin A, these equations will become

$$x = r \cos. v,$$
$$y = r \sin. v.$$

SECTION IV.

ON THE CIRCLE.

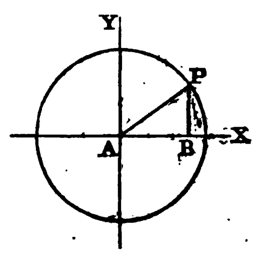
(32.) A CIRCLE is a plane figure bounded by a line, every point of which is equally distant from a point within called the center. This bounding line is called the circumference of the circle. A radius of a circle is a straight line drawn from the center to the circumference.

PROPOSITION I.—THEOREM.

(33.) *The equation of the circle, when the origin of co-ordinates is at the center, is*

$$x^2+y^2=R^2;$$

where R is the radius of the circle, and x and y the co-ordinates of any point of the circumference.



Let A be the center of the circle; it is required to find the equation of a curve such that every point of it shall be equally distant from A. Represent this distance by R, and let x and y represent the co-ordinates of any point of the curve, as P. Then, by Geometry, Prop. 11, B. IV.,

$$AB^2+BP^2=AP^2;$$

that is, $$x^2+y^2=R^2,$$

which is the equation required.

(34.) If we wish to determine the points where the curve cuts the axis of X, we must make

$$y=0;$$

for this is the property of all points situated on the axis of abscissas. On this supposition we have

$$x=\pm R;$$

which shows that the curve cuts the axis of abscissas in two points on different sides of the origin, and at a distance from it equal to the radius of the circle.

To determine the points where the curve cuts the axis of ordinates, we make $x=0$, and we obtain

$$y=\pm R;$$

which shows that the curve cuts the axis of ordinates in two points on different sides of the origin, and at a distance from it equal to the radius of the circle.

(35.) If we wish to trace the curve through the intermediate points, we reduce the equation to the form

$$y=\pm\sqrt{R^2-x^2}.$$

Now, since every value of x furnishes two equal values of y, with contrary signs, it follows that the curve is symmetrical above and below the axis of X.

If we suppose x to be positive, the values of y continually decrease from $x=0$, which gives $y=\pm R$, to $x=+R$, which gives $y=0$.

If we make x greater than R, y becomes imaginary, which shows that the curve does not extend on the side of the positive abscissas beyond the value of $x=+R$.

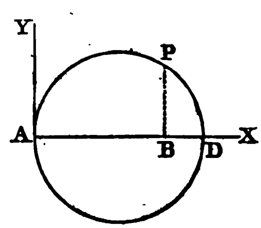
In the same manner it may be shown that the curve does not extend on the side of the negative abscissas beyond the value of $x=-R$.

PROPOSITION II.—THEOREM.

(36.) *The equation of the circle, when the origin of co-ordinates is on the circumference, is*

$$y^2=2Rx-x^2,$$

where R is the radius of the circle, and x and y the co-ordinates of any point of the circumference.



Let the origin of co-ordinates be at A, a point on the circumference of the circle. Draw AX, the axis of abscissas, through the center of the circle. Let P be any point on the circumference, and draw PB perpendicular to AX. Denote the line AD by 2R, the distance AB by x, and the perpendicular BP by y; then BD will be represented by $2R-x$.

Now BP is a mean proportional between the segments AB and BD (Geom., Prop. 22, Cor., B. IV.); that is,

$$BP^2=AB\times BD,$$

or $$y^2=x(2R-x)=2Rx-x^2,$$

which is the equation required.

(37.) If we wish to determine where the curve cuts the axis of X, we make $y=0$, and we obtain

$$x(2R-x)=0.$$

This equation is satisfied by supposing $x=0$, or $2R-x=0$, from the last of which equations we derive $x=2R$. The curve, therefore, cuts the axis of abscissas in two points, one at the origin, and the other at a distance from it equal to 2R.

To determine where the curve cuts the axis of ordinates, we make $x=0$, which gives

$$y=0,$$

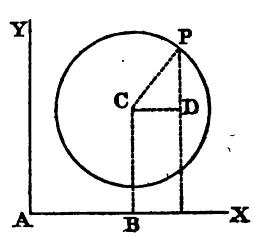
which shows that the curve meets the axis of ordinates in but one point, viz., the origin.

PROPOSITION III.—THEOREM.

(38.) *The most general equation of the circle is*

$$(x-x')^2+(y-y')^2=R^2,$$

where R denotes the radius of the circle, x' and y' are the co-ordinates of the center, and x and y the co-ordinates of any point of the circumference.



Let C be the center of the circle, and assume any rectangular axes AX, AY. Let the co-ordinates AB, BC of the center be denoted by x' and y'; while the co-ordinates of any point P in the circumference are denoted by x and y. Then, if we draw the radius CP, and CD parallel to the axis of X, we shall have

$$CD=x-x',$$

and $$PD=y-y'.$$

But $$CD^2+PD^2=CP^2.$$

Hence we have $$(x-x')^2+(y-y')^2=R^2,$$

which is the equation sought.

(39.) To find the points where the curve intersects the axis of X, we must make $y=0$, which gives,

$$(x-x')^2+y'^2=R^2,$$

whence $$(x-x')^2=R^2-y'^2,$$
$$x-x'=\pm\sqrt{R^2-y'^2},$$
or $$x=x'\pm\sqrt{R^2-y'^2},$$
where we see that the values of x will become imaginary when y' exceeds R; and it is evident that if the distance of the center of the circle from the axis of abscissas exceeds the radius of the circle there can be no intersection.

To find the point where the curve intersects the axis of Y, we must make $x=0$, which gives

$$y=y'\pm\sqrt{R^2-x'^2},$$

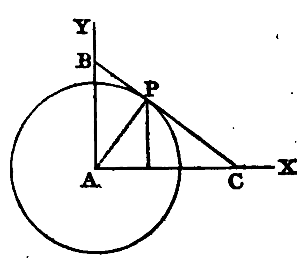
which becomes imaginary when x' exceeds R, and it is plain that in this case there can be no intersection.

PROPOSITION IV.—THEOREM.

(40.) *The equation of a tangent line to the circle is*

$$xx'+yy'=R^2,$$

where R denotes the radius of the circle, x' and y' are the co-ordinates of the point of contact, and x and y are the general co-ordinates of the tangent line.



Let BC be a line touching the circle, whose center is A, in the point P. Let the co-ordinates of the point P be x' and y', and draw the radius AP. The equation of the line AP, passing through the origin and through the point x', y', Art. 21, is

$$y=\frac{y'}{x'}x.$$

Now a tangent is perpendicular to the radius at the point of contact (Geom., Prop. IX., B. III.). But the equation of a line passing through a given point, and perpendicular to a given line, Art. 25, is

$$y-y'=-\frac{1}{a'}(x-x').$$

The value of a', taken from the equation of the radius, is

$$\frac{y'}{x'}.$$

Hence $$-\frac{1}{a'}=-\frac{x'}{y'}.$$

The equation of the tangent line is, therefore,

$$y-y'=-\frac{x'}{y'}(x-x').$$

Clearing of fractions and transposing, we obtain

$$xx'+yy'=x'^2+y'^2.$$

But since the point P is on the circumference, its co-ordinates must satisfy the equation of the circle; that is,

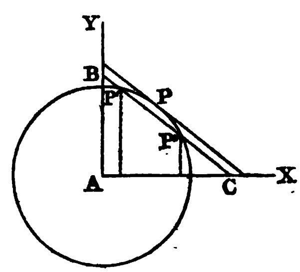
$$x'^2+y'^2=R^2.$$

Hence $$xx'+yy'=R^2,$$

which is the equation required.

(41.) The equation of the tangent may also be obtained, without employing the geometrical property above referred to, by a method which is applicable to all curves whatever.

Let us first consider a line BC, meeting the curve in two points P′ and P″; the co-ordinates of P′ being represented by x', y', and those of P″ by x'', y''. The equation of the line BC, Art. 20, is



$$y-y'=\frac{y'-y''}{x'-x''}(x-x'); \quad (1)$$

and, since both the points P′ and P″ are on the circumference, we must have

$$x'^2+y'^2=R^2, \quad (2)$$

and $$x''^2+y''^2=R^2. \quad (3)$$

Subtracting equation (3) from equation (2), we obtain

$$y'^2-y''^2+x'^2-x''^2=0;$$

that is, $$(y'+y'')(y'-y'')+(x'+x'')(x'-x'')=0.$$

whence $$\frac{y'-y''}{x'-x''}=-\frac{x'+x''}{y'+y''}.$$

Substituting this value in equation (1), we obtain

$$y-y'=-\frac{x'+x''}{y'+y''}(x-x'). \quad (4)$$

If now we suppose the secant BC to move toward the point P, the point P′ will approach P″; and when P′ coincides with P″, the secant line will become a tangent to the circumference.

When this takes place, x' will equal x'', and y' will equal y'', and the last equation becomes

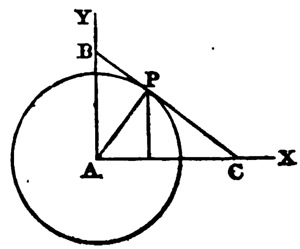
$$y-y'=-\frac{x'}{y'}(x-x'),$$

as before found.

(42.) To determine the point in which the tangent intersects the axis of X, we make $y=0$, which gives

$$xx'=R^2,$$

or $$x=\frac{R^2}{x'}=AC.$$



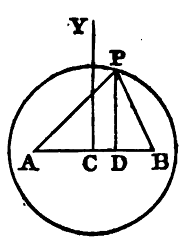
To determine the point in which the tangent intersects the axis of Y, we make $x=0$, which gives

$$yy'=R^2,$$

or $$y=\frac{R^2}{y'}=AB.$$

PROPOSITION V.—PROBLEM.

(43.) *Given the base of a triangle, and the sum of the squares of its sides, to determine the triangle.*



Let AB be the base of the proposed triangle. Bisect AB in C; draw CY perpendicular to AB, and assume YC, CB as a system of rectangular axes.

Let x and y be the co-ordinates of P, the vertex of the triangle, and from P let fall the perpendicular PD. Let a denote AC or CB, and put m for the sum of the squares of the sides AP, BP.

Then, by Geom., Prop. XI., B. IV., we shall have

$$PD^2+AD^2=AP^2,$$

and $$PD^2+BD^2=BP^2;$$

or $$y^2+(x+a)^2=AP^2,$$

and $$y^2+(x-a)^2=BP^2.$$

Adding these equations together, we obtain

$$2y^2+2x^2+2a^2=AP^2+BP^2=m.$$

Whence $$y^2+x^2=\frac{m}{2}-a^2.$$

Comparing this result with Art. 33, we see that this equation represents a circle whose center is the origin C, and the radius

$$\sqrt{\frac{m}{2}-a^2};$$

so that if this circle be described, and lines be drawn from A and B to any point in its circumference, a triangle will be formed which satisfies the proposed conditions.

Proposition VI.—Theorem.

(44.) *The polar equation of the circle, when the origin is on the circumference, is*

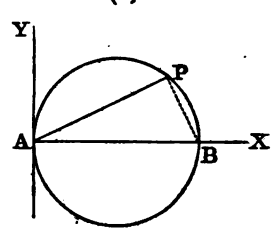
$$r=2\text{R} \cos. v,$$

where R represents the radius of the circle, r the radius vector, and v the variable angle.

The equation of the circle referred to rectangular axes, when the origin is on the circumference, Art. 36, is

$$y^2=2\text{R}x-x^2. \qquad (1)$$

Let A be the position of the pole, and AX the line from which the variable angle is estimated. The formulas for passing from a system of rectangular to a system of polar co-ordinates, the origin remaining the same, Art. 31, are



$$x=r \cos. v,$$
$$y=r \sin. v.$$

Squaring each member of these equations, and substituting the values of x^2, y^2, thus found in equation (1), we obtain

$$r^2 \sin.^2 v=2\text{R}r \cos. v-r^2 \cos.^2 v;$$

or, by transposition, $r^2(\sin.^2 v+\cos.^2 v)=2\text{R}r \cos. v.$

But $\sin.^2 v+\cos.^2 v$ is equal to unity.

Hence $$r^2=2\text{R}r \cos. v;$$

or, dividing by r, we obtain

$$r=2\text{R} \cos. v,$$

which is the polar equation of the circle.

(45.) This equation might have been derived directly from the figure. Thus, by Trig., Art. 41,

$$\text{radius} : AB :: \cos. BAP : AP,$$

or
$$1 : 2R :: \cos. \quad v \quad : r;$$

whence
$$r = 2R \cos. v.$$

(46.) When $v=0$, the cos. $v=1$, and we have

$$r = 2R = AB.$$

As v increases from 0 to 90°, the radius vector determines all the points in the semicircumference BPA; and when $v=90°$, then cos. $v=0$, and we have

$$r = 0.$$

From $v=270°$ to $v=360°$, the radius vector will determine all the points of the semicircumference below the axis of abscissas.

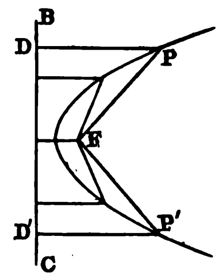
SECTION V.

ON THE PARABOLA.

(47.) A PARABOLA is a plane curve, every point of which is equally distant from a fixed point and a given straight line.

The fixed point is called the *focus* of the parabola, and the given straight line is called the *directrix*.

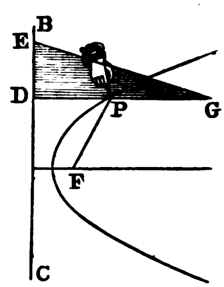
Thus, if F be a fixed point, and BC a given line, and the point P move about F in such a manner that its distance from F is always equal to the perpendicular distance from BC, the point P will describe a parabola of which F is the focus and BC the directrix.



The distance of any point of the curve from the focus, is called the radius vector of that point.

(48.) From the definition of the parabola, the curve may be described mechanically.

Let BC be a ruler laid upon a plane, and let DEG be a square. Take a thread equal in length to DG, and attach one extremity at G, and the other at some point, as F. Then slide the side of the square DE along the ruler BC, and at the same time keep the thread continually tight by means of the pencil P; the pencil will describe one part of a parabola, of which F is the focus, and BC the directrix. For in every position of the square,



$$PF+PG=PD+PG,$$

and hence $$PF=PD;$$

that is, the point P is equally distant from the focus F and the directrix BC.

If the square be turned over and moved on the other side of

the point F, the other part of the same parabola may be described.

(49.) A *diameter* is a straight line drawn through any point of the curve perpendicular to the directrix. The *vertex* of the diameter is the point in which it cuts the curve.

The *axis* of the parabola is the diameter which passes through the focus.

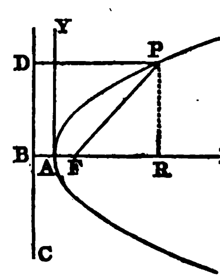
The *parameter* of a diameter is the double ordinate which passes through the focus.

PROPOSITION I.—THEOREM.

(50.) *The equation of the parabola, referred to rectangular axes whose origin is at the vertex of the axis, is*

$$y^2=2px,$$

where x and y are the general co-ordinates of the curve, and $2p$ is the parameter of the axis.



Let F be the focus, and DC the directrix. Take AX as the axis of abscissas, and let the origin be placed at A, the middle point of BF. Represent BF by p, whence AF will equal $\frac{p}{2}$. Let x and y be the co-ordinates of any point P in the curve, and represent FP by r.

By the definition of the curve,

$$PF=PD=AR+AB=x+\frac{p}{2}.$$

Also, $$FR=x-\frac{p}{2}.$$

But $$PR^2+FR^2=PF^2;$$

that is, $$y^2+\left(x-\frac{p}{2}\right)^2=\left(x+\frac{p}{2}\right)^2.$$

Whence, by expanding, we obtain

$$y^2=2px.$$

(51.) *Cor.* 1. If we make $x=0$, we have

$$y=0,$$

which shows that the curve passes through the origin A.

If we make $x=\frac{p}{2}$, we shall have

$$y^2=p^2,$$

or $$y=p;$$

whence $$2y=2p;$$

that is, the constant quantity $2p$, called the parameter, is equal to the double ordinate through the focus, conformably to the definition, Art. 49.

Cor. 2. From the equation of the parabola we obtain

$$y=\pm\sqrt{2px},$$

which shows that for every value of x there will be two equal values of y, with contrary signs. Hence the curve is symmetrical with respect to the axis of X.

Cor. 3. If we convert the equation $y^2=2px$ into a proportion, we shall have

$$x:y::y:2p;$$

that is, *the parameter of the axis is a third proportional to any abscissa and its corresponding ordinate.*

Cor. 4. *The squares of ordinates to the axis are to each other as their corresponding abscissas.*

Designate any two ordinates by y', y'', and the corresponding abscissas by x', x'', then we shall have

$$y'^2=2px',$$

and $$y''^2=2px''.$$

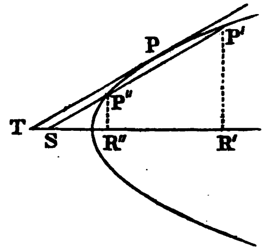
Hence $$y'^2:y''^2::2px':2px''::x':x''.$$

PROPOSITION II.—THEOREM.

(52.) *The equation of a tangent line to the parabola is*

$$yy'=p(x+x'),$$

where x', y' are the co-ordinates of the point of contact, and p is half the parameter of the axis.



Draw any line P'P'', cutting the parabola in the points P', P''; if this line be moved toward P, it will approach the position of the tangent, and the secant will become a tangent when the points P', P'' coincide.

Let x', y' be the co-ordinates of the point P', and x'', y'' the co-ordinates of the point P''. The equation of the line passing through these two points, Art. 20, will be

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'), \qquad (1)$$

Since the points P′, P″ are on the curve. we shall have

$$y'^2=2px', \qquad (2)$$
$$y''^2=2px''. \qquad (3)$$

These three equations express three conditions which the secant line must fulfill, and if we combine them in one we shall obtain a single equation from which two of the unknown quantities have been eliminated; and we are at liberty to combine them together, because each of these equations must be satisfied at the same time.

Subtracting equation (3) from (2), we have

$$y'^2-y''^2=2p(x'-x'').$$

Whence $$\frac{y'-y''}{x'-x''}=\frac{2p}{y'+y''}.$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y-y'=\frac{2p}{y'+y''}(x-x'). \qquad (4)$$

The secant will become a tangent when the points P′, P′ coincide, in which case

$$x'=x'' \text{ and } y'=y''.$$

Equation (4) in this case becomes

$$y-y'=\frac{p}{y'}(x-x'),$$

which is the equation of a tangent to the parabola at the point P. If we clear this equation of fractions, we have

$$yy'-y'^2=px-px'.$$

But $$y'^2=2px'.$$

Hence $$yy'=px-px'+2px',$$

or $$yy'=p(x+x').$$

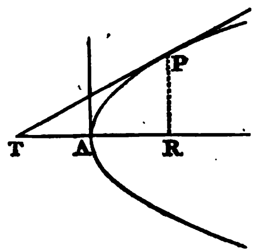
(53.) DEFINITION. A *subtangent* is that part of a diameter intercepted between a tangent and ordinate to the point of contact.

Cor. 1. To find the point in which the tangent intersects the axis of abscissas, make $y=0$ in the equation of the tangent, and we have

$$0=p(x+x');$$

that is, $$x=-x',$$
or $$AR=-AT;$$
that is, *the subtangent is bisected at the vertex.*

Cor. 2. This property enables us to draw a tangent to the curve through a given point. Let P be the given point; from P draw PR perpendicular to the axis, and make AT=AR. Draw a line through P and T, and it will be a tangent to the parabola at P.



Cor. 3. In the equation

$$y-y'=\frac{p}{y'}(x-x'),$$

$\frac{p}{y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the axis of the parabola.

(54.) Definitions. A *normal* is a line drawn perpendicular to a tangent from the point of contact, and terminated by the axis.

A *subnormal* is the part of the axis intercepted between the normal and the corresponding ordinate.

PROPOSITION III.—THEOREM.

(55.) *The equation of a normal line to the parabola is*

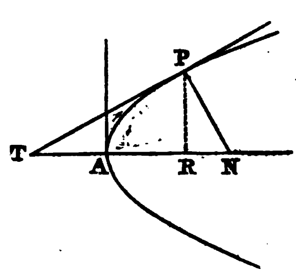
$$y-y'=-\frac{y'}{p}(x-x'),$$

where x', y' are the co-ordinates of the point of intersection with the curve.

The equation of a straight line passing through the point whose co-ordinates are x', y', Art. 18, is

$$y-y'=a(x-x'); \qquad (1)$$

and, since the normal line is perpendicular to the tangent, we shall have, Art. 23,



$$a=-\frac{1}{a'}.$$

But we have found for the tangent line, Prop. II., Cor. 3,

$$a'=\frac{p}{y'}.$$

Hence $$a=-\frac{y'}{p}.$$

Substituting this value in equation (1), we shall have for the equation of the normal line

$$y-y'=-\frac{y'}{p}(x-x'). \qquad (2)$$

(56.) *Cor.* To find the point in which the normal intersects the axis of abscissas, make $y=0$ in equation (2), and we have, after reduction,

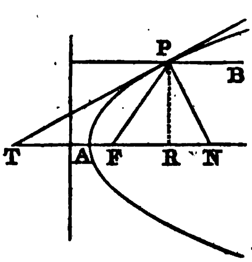
$$x-x'=p.$$

But x is equal to the distance AN, and x' to AR; hence $x-x'=p$ is equal to RN; that is, *the subnormal is constant, and equal to half the parameter of the axis.*

PROPOSITION IV.—THEOREM.

(57.) *The normal, at any point of the parabola, bisects the angle made by the radius vector and the diameter passing through that point.*

Let PT be a tangent to a parabola, PF the radius vector, PN the normal, and PB the diameter to the point P; the normal PN bisects the angle BPF. Let x' represent the abscissa of the point P.



Now FN=AR+RN−AF.

But AR$=x'$, RN$=p$, and AF$=\frac{p}{2}$.

Hence $$\text{FN}=x'+p-\frac{p}{2}=x'+\frac{p}{2}.$$

But in Prop. I. we found

$$\text{FP}=x'+\frac{p}{2}.$$

Hence $$\text{FN}=\text{FP}.$$

Therefore tne angle FPN=FNP=the alternate angle BPN.

(58.) *Cor.* $$\text{FR}=\text{AR}-\text{AF}=x'-\frac{p}{2}.$$

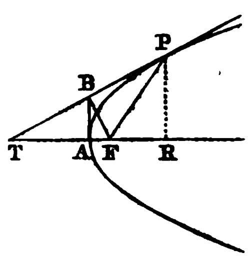
But TR$=2x'$, Prop. II., Cor. 1.

Hence $\quad TF=TR-FR=x'+\frac{p}{2}=PF;$

that is, *if a tangent to the parabola cut the axis produced, the points of contact and of intersection are equally distant from the focus.*

Proposition V.—Theorem.

(59.) *If a perpendicular be drawn from the focus to any tangent, the perpendicular will be a mean proportional between the distances of the focus from the vertex and from the point of contact.*



Let FB be a perpendicular drawn from the focus to the tangent PT. Join AB, and draw the ordinate PR.

Since FT is equal to FP (Prop. IV., Cor.), and FB is drawn perpendicular to PT, PB is equal to BT. But RA is equal to AT, Prop. II., Cor. 1; hence

TB : BP :: TA : AR,

and therefore AB is parallel to PR. But PR is perpendicular to the axis; hence AB is perpendicular to TF; and therefore, by similar triangles, FAB, FBT, we have

FA : FB :: FB : FT or FP.

Proposition VI.—Theorem.

(60.) *The equation of the parabola referred to a tangent line, and the diameter passing through the point of contact, the origin being the point of contact, is*

$$y^2=2p'x,$$

where $2p'$ is the parameter of the diameter passing through the origin.

The formulas for passing from rectangular to oblique axes are (Art. 30, Schol.),

$$x=a+x'\cos.\alpha+y'\cos.\alpha', \qquad (1)$$
$$y=b+x'\sin.\alpha+y'\sin.\alpha'. \qquad (2)$$

Since the new origin is to be on the curve, its co-ordinates must satisfy the equation of the curve; that is,

$$b^2=2pa, \text{ whence } a=\frac{b^2}{2p}.$$

Also, since every diameter is parallel to the axis, we must have

$$\alpha=0;$$

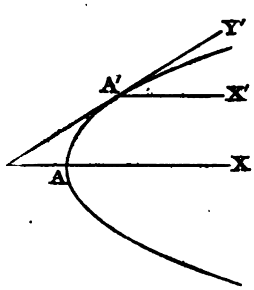
whence $\qquad \sin.\ \alpha=0$, and $\cos.\ \alpha=1$.

And, since the tangent of the angle which a tangent line makes with the axis of the parabola (Prop. II., Cor. 3) is $\frac{p}{y'}$, we must have

$$\frac{p}{y'} \text{ or } \frac{p}{b}=\text{tang. } \alpha'=\frac{\sin.\ \alpha'}{\cos.\ \alpha'};$$

whence $$\cos.\ \alpha'=\frac{b \sin.\ \alpha'}{p}.$$

Making these substitutions, formulas (1) and (2) become



$$x=\frac{b^2}{2p}+x'+\frac{by' \sin.\ \alpha'}{p},$$

and $$y=b+y' \sin.\ \alpha'.$$

Substituting these values in the general equation of the parabola

$$y^2=2px,$$

we have

$$b^2+2by' \sin.\ \alpha'+y'^2 \sin.^2 \alpha'=b^2+2px'+2by' \sin.\ \alpha';$$

or $$y'^2 \sin.^2 \alpha'=2px';$$

whence $$y'^2=\frac{2px'}{\sin.^2 \alpha'}.$$

If we put $p'=\frac{p}{\sin.^2 \alpha'}$, and omit the accents of the variables we shall have

$$y^2=2p'x,$$

which is the equation required; where $2p'$ is called the parameter of the diameter A′X′. See Art. 63.

(61.) *Cor. The squares of ordinates to any diameter are to each other as their corresponding abscissas.*

Designate any two ordinates by y', y'', and the corresponding abscissas by x', x'', we shall have

$$y'^2=2p'x',$$

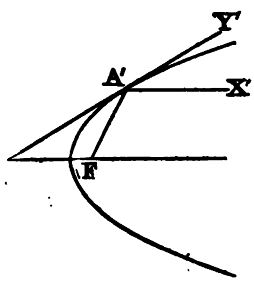
and $$y''^2=2p'x''.$$

Hence $$y'^2 : y''^2 :: 2p'x' : 2p'x'' :: x' : x''.$$

PROPOSITION VII.—THEOREM.

(62.) *The parameter of any diameter is equal to four times the distance from the vertex of that diameter to the focus.*

We have from the last Proposition,



$$\frac{p}{b}=\frac{\sin.\ \alpha'}{\cos.\ \alpha'},$$

whence $\quad b \sin.\ \alpha' = p \cos.\ \alpha',$

and $\quad b^2 \sin.^2\ \alpha' = p^2 \cos.^2\ \alpha',$

$$=p^2(1-\sin.^2\ \alpha'),$$

$$=p^2-p^2 \sin.^2\ \alpha'.$$

Therefore $\sin.^2\ \alpha'=\frac{p^2}{b^2+p^2}.$

But $b^2=2ap$, from the equation of the curve.

Hence $$\sin.^2\ \alpha'=\frac{p^2}{2ap+p^2}=\frac{p}{2a+p}.$$

Now p' was taken equal to $\frac{p}{\sin.^2\ \alpha'}$ (Art. 60).

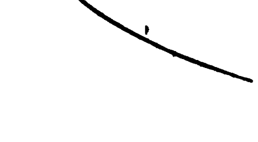
Hence $$p'=2a+p,$$

and $$2p'=4\left(a+\frac{p}{2}\right).$$

But $a+\frac{p}{2}$ is equal to A′F (Prop. I.).

Hence $2p'$, or the parameter of the diameter A′X′, is equal to 4A′F.

(63.) SCHOLIUM. If through the focus F the line BD be drawn parallel to the tangent TA′, then calling x and y the co-ordinates of the point D,



$$x=\text{A}'\text{C}=\text{TF}=\text{A}'\text{F} \text{ (Prop. IV., Cor.)},$$

$$=\frac{p'}{2} \text{ (Prop. VII.)}.$$

But, by Prop. VI., $y^2=2p'x$.

Hence $$y^2=2p'\times\frac{p'}{2}=p'^2,$$

or $$y=p',$$

and $$2y=2p';$$

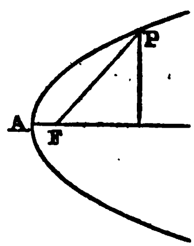
that is, the quantity $2p'$, which has been called the parameter of the diameter A′X′, is equal to the double ordinate passing through the focus, conformably to the definition, Art. 49.

PROPOSITION VIII.—THEOREM.

(64.) *The polar equation of the parabola, the pole being at the focus, is*

$$r=\frac{p}{1+\cos. v},$$

where p represents half the parameter, and v is the angle which the radius vector makes with the axis.



We have found the distance of any point of the parabola from the focus, Prop. I., to be

$$r=\text{FP}=x+\frac{p}{2},$$

where the abscissa x is reckoned from the vertex A. In order to transfer the origin from A to F, we must substitute for x, $x'+\frac{p}{2}$;

whence $$r=x'+p.$$

If we represent the angle PFA by v, we shall have (Trig., Art. 41)

$$x'=-r\cos. v;$$

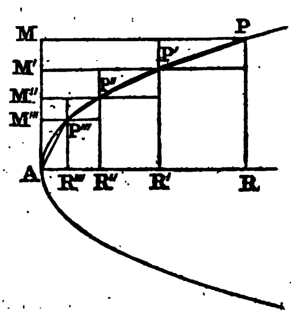
whence $$r=p-r\cos. v,$$

or $$r=\frac{p}{1+\cos. v},$$

where the angle v is estimated from the vertex A toward the right.

PROPOSITION IX.—THEOREM.

(65.) *The area of any segment of a parabola is equal to two thirds of the rectangle described on its abscissa and ordinate.*



Let APR be a segment of a parabola bounded by the axis AR and the ordinate PR. Complete the rectangle AMPR; then will the parabolic segment APR be two thirds of the rectangle AMPR.

Inscribe in the parabola a polygon PP′P″…AR, and through the points P, P′, P″, etc., draw parallels to AR and PR, forming the interior rectan-

gles P′R, P″R′, etc., and the corresponding exterior rectangles P′M, P″M′, etc. Designate the former by P, P′, P″, etc.; the latter by p, p', p'', etc., and the corresponding co-ordinates by x, y, x', y', etc., we shall then have

$$P'R = P'R' \times R'R,$$

or

$$P = y'(x - x').$$

Also,

$$P'M = P'M' \times MM',$$

or

$$p = x'(y - y');$$

which gives

$$\frac{P}{p} = \frac{y'(x - x')}{x'(y - y')}. \qquad (1)$$

But, since the points P, P′, etc., are on the curve, we have

$$y^2 = 2px, \quad y'^2 = 2px';$$

whence

$$x - x' = \frac{y^2 - y'^2}{2p}, \text{ and } x' = \frac{y'^2}{2p}.$$

Substituting these values in equation (1), we obtain

$$\frac{P}{p} = \frac{y'(y^2 - y'^2)}{y'^2(y - y')} = \frac{y + y'}{y'} = 1 + \frac{y}{y'}.$$

In the same manner we find

$$\frac{P'}{p'} = 1 + \frac{y'}{y''},$$

$$\frac{P''}{p''} = 1 + \frac{y''}{y'''}, \text{ etc.}$$

If, now, we suppose the vertices of the polygons P, P′, P″, etc., to be so placed that the ordinates shall be in geometrical progression, we shall have

$$\frac{y}{y'} = \frac{y'}{y''} = \frac{y''}{y'''}, \text{ etc.},$$

so that each interior rectangle has to its corresponding exterior rectangle the ratio of $1 + \frac{y}{y'}$ to 1.

Therefore, by composition,

$$\frac{P + P' + P'' +, \text{ etc.},}{p + p' + p'' +, \text{ etc.},} = 1 + \frac{y}{y'};$$

that is, the sum of all the interior rectangles is to the sum of all the exterior rectangles as $1 + \frac{y}{y'}$ to 1.

The nearer the points P, P′, P″ are taken to each other, the nearer does the sum of the interior rectangles approach to the

area of the parabolic segment, and the ratio $\frac{y}{y'}$ approaches to a ratio of equality. Hence, designating the area APR by S, and the area AMP by s, we have

$$\frac{S}{s}=1+1=2.$$

or $$\frac{S+s}{s}=3;$$

whence $$S=\frac{2}{3}(S+s).$$

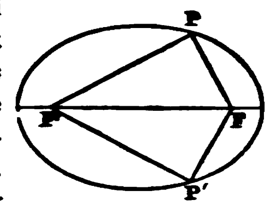
But $S+s$ is equal to the area of the rectangle AMPR; hence the parabolic segment is two thirds of the area of the circumscribing rectangle.

SECTION II.

ON THE ELLIPSE.

(66.) An ellipse is a plane curve in which the sum of the distances of each point from two fixed points is equal to a given line. The two fixed points are called the *foci*.

Thus, if F and F′ are two fixed points, and if the point P moves about F in such a manner that the sum of its distances from F and F′ is always the same, the point P will describe an ellipse, of which F and F′ are the foci. The distance of the point P from either focus is called the *radius vector*.



(67.) From the definition of an ellipse, the curve may be described mechanically. Thus, take a thread longer than the distance FF′, and fasten one of its extremities at F, the other at F′. Then let a pencil be made to glide along the thread, so as to keep it always stretched; the curve described by the point of the pencil will be an ellipse.

The *center* of the ellipse is the middle point of the straight line joining the foci.

A *diameter* is a straight line drawn through the center and terminated both ways by the curve.

The *major axis* is the diameter which passes through the foci. The *minor axis* is the diameter which is perpendicular to the major axis.

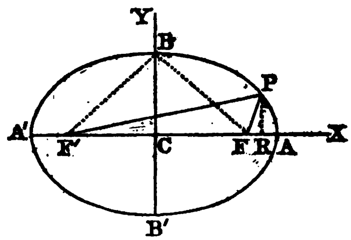
The *parameter* of the major axis is the double ordinate which passes through one of the foci.

Proposition I.—Theorem.

(68.) *The equation of the ellipse, referred to its center and axes, is*

$$A^2y^2+B^2x^2=A^2B^2,$$

where A and B represent the semi-axes, and x and y the general co-ordinates of the curve.

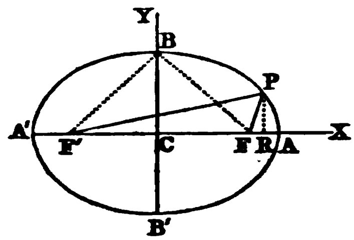


Let F and F′ be the foci, and draw the rectangular axes CX, CY, the origin C being placed at the middle of FF′. Let P be any point of the curve, and draw PR perpendicular to CX. Let the sum of the distances of the point P from the foci be represented by 2A. Denote the distance CF or CF′ by c; FP by r, and F′P by r'; and let x and y represent the co-ordinates of the point P.

Then, since $$\text{FP}^2=\text{PR}^2+\text{RF}^2,$$
we have $$r^2=y^2+(x-c)^2. \quad (1)$$
Also, $$\text{PF}'^2=\text{PR}^2+\text{RF}'^2;$$
that is, $$r'^2=y^2+(x+c)^2. \quad (2)$$

Adding equations (1) and (2), we obtain
$$r^2+r'^2=2(y^2+x^2+c^2); \quad (3)$$
and, subtracting equation (1) from (2), we obtain
$$r'^2-r^2=4cx,$$
which may be put under the form
$$(r'+r)(r'-r)=4cx. \quad (4)$$
But we have, from the definition of the ellipse,
$$r'+r=2\text{A}.$$
Substituting this value in equation (4), we obtain
$$r'-r=\frac{2cx}{\text{A}}.$$
Combining the last two equations, we find
$$r'=\text{A}+\frac{cx}{\text{A}}, \quad (5)$$
and
$$r=\text{A}-\frac{cx}{\text{A}}. \quad (6)$$
Squaring these values, and substituting them in equation (3), we obtain
$$\text{A}^2+\frac{c^2x^2}{\text{A}^2}=y^2+x^2+c^2,$$
which may be reduced to
$$\text{A}^2y^2+(\text{A}^2-c^2)x^2=\text{A}^2(\text{A}^2-c^2), \quad (7)$$
which is the equation of the ellipse.

This equation may, however, be put under a more convenient form. Represent the line BC by B. In the two right-angled triangles BCF, BCF′, CF is equal to CF′, and BC is common to both triangles; hence BF is equal to BF′. But BF +BF′, by the definition of the ellipse, is equal to 2A; consequently BF is equal to A.



Now $\qquad BC^2=BF^2-FC^2$:

that is, $\qquad B^2=A^2-c^2. \qquad (8)$

Substituting this value in equation (7), we obtain

$$A^2y^2+B^2x^2=A^2B^2,$$

which is the equation required.

(69.) SCHOLIUM. Transposing, and dividing by A^2, this equation reduces to $y^2=\frac{B^2}{A^2}(A^2-x^2)$.

Cor. 1. To determine where the curve intersects the axis of abscissas, make $y=0$ in the equation of the ellipse, and we obtain

$$x=\pm A=CA \text{ or } CA',$$

which shows that the curve cuts the axis of X in two points, A and A′, at the same distance from the origin, the one being to the right, the other to the left; and, since 2CA or AA′ is equal to 2A, it follows that *the sum of the two lines, drawn from any point of an ellipse to the foci, is equal to the major axis.*

Cor. 2. If we make $x=0$, in the equation of the ellipse, we obtain

$$y=\pm B=CB \text{ or } CB',$$

which shows that the curve cuts the axis of Y in two points, B and B′, at the same distance from the origin.

Cor. 3. When B is made equal to A, the equation of the ellipse becomes

$$y^2+x^2=A^2,$$

which is the equation of a circle; hence the ellipse becomes a circle when its axes are made equal to each other.

Cor. 4. Since BF or BF′ is equal to A, it follows that *the*

distance from either focus to the extremity of the minor axis, is equal to half the major axis.

Cor. 5. According to the Scholium, Art. 69,

$$y^2=\frac{B^2}{A^2}(A^2-x^2).$$

Suppose $x=c$, or CF, then

$$y^2=\frac{B^2}{A^2}(A^2-c^2).$$

But by Art. 68, Equation (8),

$$A^2-c^2=B^2.$$

Hence $$y^2=\frac{B^2}{A^2}\times B^2,$$

or $$A : B :: B : y,$$

and $$2A : 2B :: 2B : 2y.$$

But $2y$ represents the double ordinate drawn through the focus, and is called the parameter, Art. 67; hence *the parameter is a third proportional to the major and minor axes.*

Cor. 6. The quantity $\frac{c}{A}$, or the distance from the center to either focus, divided by the semi-major axis, is called the *eccentricity* of the ellipse. If we represent the eccentricity by e, then

$$\frac{c}{A}=e, \text{ or } c=Ae.$$

But we have seen that $c^2=A^2-B^2$.

Hence $$A^2-B^2=A^2e^2,$$

or $$\frac{B^2}{A^2}=1-e^2.$$

Making this substitution, the equation of the ellipse becomes

$$y^2=(1-e^2)(A^2-x^2).$$

Cor. 7. Equations (5) and (6) of the preceding Proposition are

$$r'=A+\frac{cx}{A},$$

$$r=A-\frac{cx}{A}.$$

Substituting e for $\frac{c}{A}$, these equations become

$$r' = A + ex,$$
$$r = A - ex,$$

which equations represent the distance of any point of the ellipse from either focus.

Multiplying these values together, we obtain

$$rr' = A^2 - e^2x^2,$$

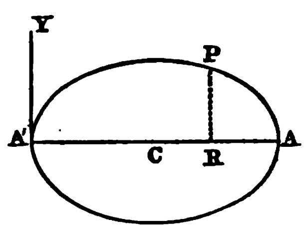
which is the value of the product of the focal distances.

PROPOSITION II.—THEOREM.

(70.) *The equation of the ellipse, when the origin is at the vertex of the major axis, is*

$$y^2 = \frac{B^2}{A^2}(2Ax - x^2),$$

where A and B represent the semi-axes, and x and y the general co-ordinates of the curve.



The equation of the ellipse, when the origin is at the center, is

$$A^2y^2 + B^2x^2 = A^2B^2. \quad (1)$$

If the origin is placed at A′, the ordinates will have the same value as when the origin was at the center, but the abscissas will be different.

If we represent the abscissas reckoned from A′ by x', then it is plain that we shall have

$$CR = A'R - A'C,$$

or
$$x = x' - A.$$

Substituting this value of x in equation (1), we have

$$A^2y^2 + B^2x'^2 - 2AB^2x' = 0,$$

which may be put under the form

$$y^2 = \frac{B^2}{A^2}(2Ax' - x'^2);$$

or, omitting the accents,

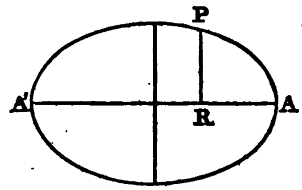
$$y^2 = \frac{B^2}{A^2}(2Ax - x^2),$$

which is the equation of the ellipse referred to the vertex of the major axis as the origin of co-ordinates.

PROPOSITION III.—THEOREM.

(71.) *The square of any ordinate is to the product of the parts into which it divides the major axis, as the square of the minor axis is to the square of the major axis.*

The equation of the ellipse, referred to the vertex A′ as the origin of co-ordinates, is, Art. 70,



$$y^2=\frac{B^2}{A^2}(2A-x)x.$$

This equation may be resolved into the proportion

$$y^2 : (2A-x)x :: B^2 : A^2.$$

Now 2A represents the major axis AA′, and, since x represents A′R, $2A-x$ will represent AR; therefore $(2A-x)x$ represents the product of the parts into which the major axis is divided by the ordinate PR.

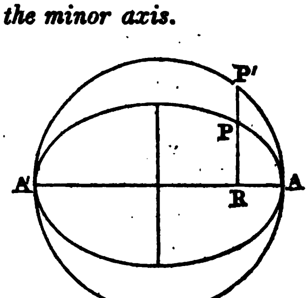
Cor. It is evident that the squares of any two ordinates are as the products of the parts into which they divide the major axis.

SCHOLIUM. It may be proved in a similar manner that the squares of ordinates to the minor axis are to each other as the products of the parts into which they divide the minor axis.

PROPOSITION IV.—THEOREM.

(72.) *If a circle be described on the major axis of an ellipse, then any ordinate in the circle is to the corresponding ordinate in the ellipse, as the major axis is to the minor axis.*

If we represent the ordinate PR in the ellipse by y', and the ordinate P′R in the circle corresponding to the same abscissa A′R by Y′, the equation of the ellipse will give us, by Art. 69,



$$y'^2=\frac{B^2}{A^2}(A^2-x^2),$$

and the equation of the circle will give, Art. 33,

$$Y'^2=(A^2-x^2).$$

Combining these two equations, we have

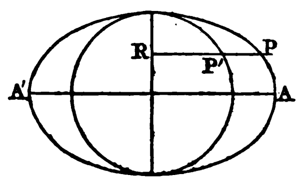
$$y'^2=\frac{B^2}{A^2}Y'^2,$$

or $$y'=\frac{B}{A}Y';$$

whence we derive the proportion

$$Y' : y' :: A : B :: 2A : 2B.$$

(73.) *Cor.* In the same manner, it may be proved that if a circle be described on the minor axis of an ellipse, any ordinate drawn to the conjugate axis is to the corresponding ordinate in the circle, as the major axis is to the minor axis.



If we represent the ordinate PR in the ellipse by x', and the corresponding ordinate P'R in the circle by X', we shall have, Prop. III., Schol.,

$$x'^2=\frac{A^2}{B^2}(B^2-y^2),$$

and $$X'^2=B^2-y^2.$$

Combining these two equations, we have

$$x'^2=\frac{A^2}{B^2}X'^2,$$

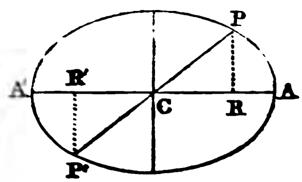
or $$x'=\frac{A}{B}X';$$

whence we derive the proportion

$$x' : X' :: A :: B :: 2A : 2B.$$

Proposition V.—Theorem.

(74.) *Every diameter of an ellipse is bisected at the center.*



Let PP' be any diameter of an ellipse. Let x', y' be the co-ordinates of the point P, and x'', y'' those of the point P'. Then, from the equation of the ellipse, we shall have, Art. 69,

$$y'^2=\frac{B^2}{A^2}(A^2-x'^2),$$

and $$y''^2=\frac{B^2}{A^2}(A^2-x''^2);$$

whence $$\frac{y'^2}{y''^2}=\frac{A^2-x'^2}{A^2-x''^2}.$$

But from the similarity of the triangles PCR, P′CR′, we have

$$\frac{y'}{y''}=\frac{x'}{x''};$$

whence $$\frac{x'^2}{x''^2}=\frac{A^2-x'^2}{A^2-x''^2}.$$

Clearing of fractions, we obtain

$$x'^2=x''^2;$$

whence, also, $$y'^2=y''^2.$$

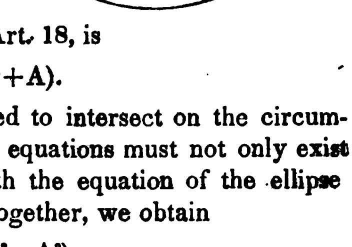
Consequently, $$x'^2+y'^2=x''^2+y''^2,$$

or $$CP^2=CP'^2;$$

that is, $$CP=CP'.$$

PROPOSITION VI.—THEOREM.

(75.) *The product of the tangents of the two angles, formed by lines drawn from the vertices of the major axis to meet in the curve, is negative, and equal to the square of the ratio of the semi-axes.*



The equation of the line AP, passing through the point A, whose co-ordinates are $x'=A$, $y'=0$, Art. 18, is

$$y=a(x-A).$$

The equation of A′P, passing through the point A′, whose co-ordinates are $x''=-A$, $y''=0$, Art. 18, is

$$y=a'(x+A).$$

Since these lines are required to intersect on the circumference of the ellipse, these two equations must not only exist at the same time, but also with the equation of the ellipse Multiplying the two equations together, we obtain

$$y^2=aa'(x^2-A^2);$$

and, since these lines intersect on the ellipse, we must have

$$y^2=\frac{B^2}{A^2}(A^2-x^2), \text{ or } -\frac{B^2}{A^2}(x^2-A^2)$$

Comparing these two equations, we perceive that

$$aa'=-\frac{B^2}{A^2}.$$

(76.) SCHOLIUM. Two lines which are drawn from the same point of a curve to the extremities of a diameter, are called *supplementary chords.*

Cor. In the circle, which may be considered an ellipse whose two axes are equal to each other, we have

$$aa'=-1,$$

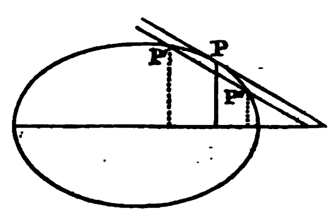
which shows that the supplementary chords are perpendicular to each other (Art. 24).

PROPOSITION VII.—THEOREM.

(77.) *The equation of a straight line which touches an ellipse is*

$$A^2yy'+B^2xx'=A^2B^2,$$

where x and y are the general co-ordinates of the tangent line, x' and y' the co-ordinates of the point of contact.



Draw any line, P'P'', cutting the ellipse in the points P', P''; if this line be moved toward P, it will approach the tangent, and the secant will become a tangent when the points P', P'' coincide.

Let x', y' be the co-ordinates of the point P', and x'', y'' the co-ordinates of the point P''. The equation of the line P'P'', passing through these two points, Art. 20, will be

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'). \qquad (1)$$

Since the points P', P'' are on the curve, we shall have

$$A^2y'^2+B^2x'^2=A^2B^2, \qquad (2)$$

and

$$A^2y''^2+B^2x''^2=A^2B^2. \qquad (3)$$

Subtracting equation (3) from (2), we have

$$A^2(y'^2-y''^2)+B^2(x'^2-x''^2)=0,$$

or

$$A^2(y'-y'')(y'+y'')=-B^2(x'-x'')(x'+x'').$$

Whence

$$\frac{y'-y''}{x'-x''}=-\frac{B^2}{A^2}\left(\frac{x'+x''}{y'+y''}\right).$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y-y'=-\frac{B^2}{A^2}\frac{x'+x''}{y'+y''}(x-x'). \qquad (4)$$

The secant P′P″ will become a tangent when the points P′, P″ coincide, in which case

$$x'=x'' \text{ and } y'=y''.$$

Equation (4), in this case, becomes

$$y-y'=-\frac{B^2}{A^2}\frac{x}{y'}(x-x'),$$

which is the equation of a tangent to the ellipse at the point P. If we clear this equation of fractions, we have

$$A^2yy'-A^2y'^2=-B^2xx'+B^2x'^2,$$

or $$A^2yy'+B^2xx'=A^2y'^2+B^2x'^2;$$

hence $$A^2yy'+B^2xx'=A^2B^2,$$

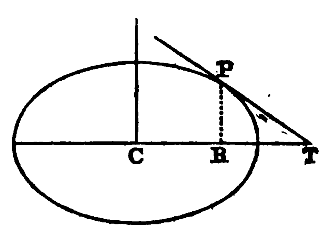
which is the most simple form of the equation of a tangent line.

(78.) *Cor.* 1. In the equation

$$y-y'=-\frac{B^2}{A^2}\frac{x'}{y'}(x-x'),$$

$-\frac{B^2}{A^2}\frac{x'}{y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the major axis.

Cor. 2. To find the point in which the tangent intersects the axis of abscissas, make $y=0$ in the equation of the tangent, and we have



$$x=\frac{A^2}{x'}.$$

which is equal to CT.

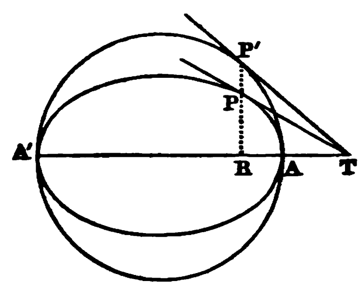
If from CT we subtract CR or x', we shall have the subtangent

$$RT=\frac{A^2}{x'}-x'=\frac{A^2-x'^2}{x'}.$$

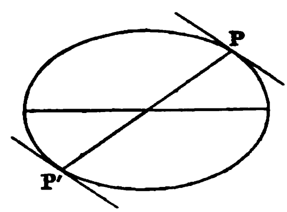
Cor. 3. This expression for the subtangent is independent of the minor axis; the subtangent is, therefore, the same for all ellipses having the same major axis; it consequently belongs to the circle described upon the major axis.

Cor. 4. Hence we are enabled to draw a tangent to an ellipse through a given point. Let P be the given point. On

AA′ describe a circle, and through P draw the ordinate PR, and produce it to meet the circumference of the circle in P′. Through P′ draw the tangent P′T, and from T, where it meets the major axis produced, draw PT; it will be a tangent to the ellipse at P.



Cor. 5. Since the co-ordinates of the point P are equal to those of the point P′, it follows from Cor. 1 that the tangents at the extremities of a diameter make equal angles with the major axis, and are therefore parallel with each other.



Hence, if tangents are drawn through the vertices of any two diameters, they will form a parallelogram circumscribing the ellipse.

PROPOSITION VIII.—THEOREM.

(79.) *The equation of a normal line to the ellipse is*

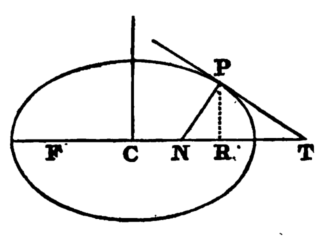
$$y-y'=\frac{A^2y'}{B^2x'}(x-x'),$$

where x and y are the general co-ordinates of the normal line, x' and y' the co-ordinates of the point of intersection with the ellipse.

The equation of a straight line passing through the point whose co-ordinates are x', y', Art. 18, is

$$y-y'=a(x-x'); \qquad (1)$$

and, since the normal line is perpendicular to the tangent, we shall have, Art. 23,



$$a=\frac{1}{-a'}.$$

But we have found for the tangent line, Prop. VII., Cor. 1,

$$a'=-\frac{B^2x'}{A^2y'}.$$

Hence $$a=\frac{A^2y'}{B^2x'}.$$

Substituting this value in equation (1), we shall have for the equation of the normal line

$$y-y'=\frac{A^2y'}{B^2x'}(x-x'). \qquad (2)$$

(80.) *Cor.* 1. To find the point in which the normal intersects the axis of abscissas, make $y=0$ in equation (2), and we have, after reduction,

$$CN=x=\frac{A^2-B^2}{A^2}x'.$$

If we subtract this value from CR, which is represented by x', we shall have the subnormal

$$NR=x'-\frac{A^2-B^2}{A^2}x'=\frac{B^2x'}{A^2}.$$

Cor. 2. If we put e^2 for $\frac{A^2-B^2}{A^2}$, Art. 69, Cor. 6, we shall have

$$CN=e^2x'.$$

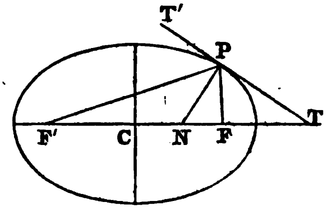
If to this we add F′C, which equals c or Ae, Prop. I., Cor. 6, we have

$$F'N=Ae+e^2x'=e(A+ex'),$$

which is the distance from the focus to the foot of the normal.

PROPOSITION IX.—THEOREM.

(81.) *The normal at any point of the ellipse bisects the angle formed by lines drawn from that point to the foci.*



Let PT be a tangent line to an ellipse, and PF, PF′ two lines drawn to the foci. Draw PN, bisecting the angle FPF′. Then, by Geometry, Prop. XVII., B. IV.,

$$FP : F'P :: FN : F'N;$$

or, by composition,

$$FP+F'P :: FF' :: F'P : F'N. \qquad (1)$$

But $$FP+F'P=2A.$$

Also, $FF'=2c=2Ae$, Prop. I., Cor. 6,

and $F'P=A+ex$, Prop. I., Cor. 7.

Making these substitutions in proportion (1), we have

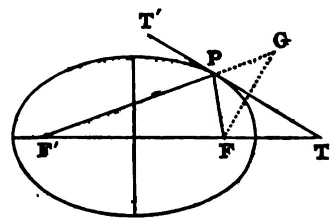
$$2A : 2Ae :: A+ex : F'N.$$

Hence $$F'N=e(A+ex).$$

But by Prop. VIII., Cor. 2, $e(A+ex)$ represents the distance from the focus F' to the foot of the normal. Hence the line PN, which bisects the angle FPF', is the normal.

(82.) *Cor.* 1. Since PN is perpendicular to TT', and the angle FPN is equal to the angle F'PN, therefore the angle FPT is equal to the angle F'PT'; that is, *the radii vectores are equally inclined to the tangent.*

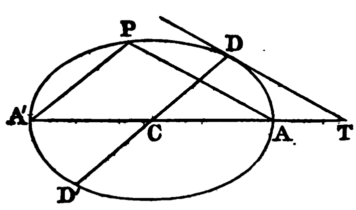
Cor. 2. This proposition affords a method of drawing a tangent line to an ellipse at a given point of the curve.



Let P be the given point; draw the radii vectores PF, PF'; produce PF' to G, making PG equal to PF, and draw FG. Draw PT perpendicular to FG, and it will be the tangent required; for the angle FPT equals the angle GPT, which equals the vertical angle F'PT'.

Proposition X.—Theorem.

(83.) *If, through one extremity of the major axis, a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter which passes through the point of contact, and conversely.*



Let DT be a tangent to the ellipse, and let the chord AP be drawn parallel to it; then will A'P be parallel to the diameter DD', which passes through the point of contact D.

Let x', y' designate the co-ordinates of D; the equation of the line CD will be, Art. 15,

$$y'=a'x';$$

whence $$a'=\frac{y'}{x'}.$$

But, by Prop. VII., Cor. 1, the tangent of the angle which the tangent line makes with the major axis, is

$$a=-\frac{B^2x'}{A^2y'}.$$

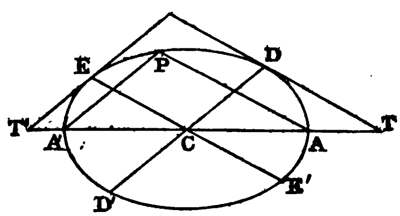
Multiplying together the values of a and a', we obtain

$$aa'=-\frac{B^2}{A^2},$$

which represents the product of the tangents of the angles DCT and DTC.

But by Prop. VI. the product of the tangents of the angles PAA′, PA′A is equal to $-\frac{B^2}{A^2}$.

Hence, if AP is parallel to DT, A′P will be parallel to CD, and conversely.



(84.) *Cor.* Let DD′ be any diameter of an ellipse, and DT the tangent drawn through its vertex, and let the chord AP be drawn parallel to DT; then, by this Proposition, the supplementary chord A′P is parallel to DD′. Let another tangent ET′ be drawn parallel to A′P, it will also be parallel to DD′. Let the diameter EE′ be drawn through the point of contact E; then, by this Proposition, A′P being parallel to T′E, AP (and, of course, DT) will be parallel to EE′. Each of the diameters DD′, EE′ is therefore parallel to a tangent drawn through the vertex of the other, and they are said to be *conjugate* to one another.

SCHOLIUM. Two diameters of an ellipse are said to be conjugate to one another, when each is parallel to a tangent line drawn through the vertex of the other.

If we designate by a and a' the tangents of the angles which two conjugate diameters make with the major axis, then we must have

$$aa'=-\frac{B^2}{A^2}.$$

PROPOSITION XI.—THEOREM.

(85.) *The equation of the ellipse, referred to its center and conjugate diameters, is*

$$A'^2y^2+B'^2x^2=A'^2B'^2;$$

where A′ and B′ are semi-conjugate diameters.

The equation of the ellipse, referred to its center and axes, Art. 65, is

$$A^2y^2+B^2x^2=A^2B^2.$$

In order to pass from rectangular to oblique co-ordinates, the origin remaining the same, we must substitute for x and y in the equation of the curve, Art. 30, the values

$$x=x' \cos. \alpha+y' \cos. \alpha',$$
$$y=x' \sin. \alpha+y' \sin. \alpha'.$$

Squaring these values of x and y, and substituting in the equation of the ellipse, we have

$$\begin{array}{l|l|l} A^2 \sin.^2 \alpha' \, y'^2 & +2A^2 \sin. \alpha \sin. \alpha' \, x'y' & +A^2 \sin.^2 \alpha' x'^2=A^2B^2; \quad (1) \\ B^2 \cos.^2 \alpha' & \quad 2B^2 \cos. \alpha \cos. \alpha' & \quad B^2 \cos.^2 \alpha \end{array}$$

which is the equation of the ellipse when the oblique co-ordinates make any angles α, α' with the major axis.

But since the new axes are conjugate diameters, we must have (Art. 84)

$$aa'=-\frac{B^2}{A^2},$$

or
$$\text{tang. } \alpha \text{ tang. } \alpha'=-\frac{B^2}{A^2};$$

whence
$$A^2 \text{ tang. } \alpha \text{ tang. } \alpha'+B^2=0.$$

Multiplying by $\quad \cos. \alpha \cos. \alpha'$,

remembering that $\quad \cos. \alpha \text{ tang.}\alpha=\sin. \alpha$,

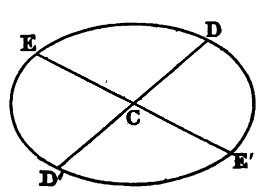
we have $\quad A^2 \sin. \alpha \sin. \alpha'+B^2 \cos. \alpha \cos. \alpha'=0.$

Hence the term containing $x'y'$ in equation (1) disappears, and we have

$$(A^2 \sin.^2 \alpha'+B^2 \cos.^2 \alpha')y'^2+(A^2 \sin.^2 \alpha+B^2 \cos.^2 \alpha)x'^2=A^2B^2, \quad (2)$$

which is the equation of the ellipse referred to conjugate diameters.

If in this equation we make $y'=0$, we shall have



$$x'^2=\frac{A^2B^2}{A^2 \sin.^2 \alpha+B^2 \cos.^2 \alpha}=CD^2.$$

If we make $x'=0$, we shall have

$$y'^2=\frac{A^2B^2}{A^2 \sin.^2 \alpha'+B^2 \cos.^2 \alpha'}=CE^2.$$

If we represent CD by A' and CE by B', equation (2) reduces to

$$\frac{y'^2}{B'^2}+\frac{x'^2}{A'^2}=1;$$

hence $$A'^2y'^2+B'^2x'^2=A'^2B'^2;$$

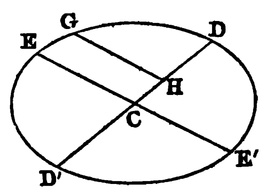
or, omitting the accents from x and y,

$$A'^2y^2+B'^2x^2=A'^2B'^2,$$

which is the equation of the ellipse referred to its center and conjugate diameters.

PROPOSITION XII.—THEOREM.

(86.) *The square of any diameter is to the square of its conjugate, as the rectangle of the parts into which it is divided by any ordinate is to the square of that ordinate.*



The equation of the ellipse, referred to conjugate diameters, is

$$A'^2y^2+B'^2x^2=A'^2B'^2,$$

which may be put under the form

$$A'^2y^2=B'^2(A'^2-x^2).$$

This equation may be reduced to the proportion

$$A'^2 : B'^2 :: A'^2-x^2 : y^2,$$

or $$(2A')^2 : (2B')^2 :: (A'+x)(A'-x) : y^2.$$

Now $2A'$ and $2B'$ represent the conjugate diameters DD′, EE′; and, since x represents CH, $A'+x$ will represent D′H, and $A'-x$ will represent DH; also, GH represents y; hence

$$DD'^2 : EE'^2 :: DH\times HD' : GH^2.$$

(87.) *Cor.* It is evident that the squares of any two ordinates to the same diameter, are as the products of the parts into which they divide that diameter.

SCHOLIUM. The *parameter* of any diameter is a third proportional to that diameter and its conjugate.

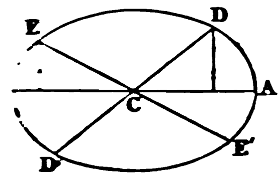
The parameter of the major axis is equal to $\frac{2B^2}{A}$, Art. 69, Cor. 5, and that of the minor axis to $\frac{2A^2}{B}$.

PROPOSITION XIII.—THEOREM.

(88.) *The sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes.*

Let DD′, EE′ be any two conjugate diameters. Designate

the coordinates of D by x', y', those of E by x'', y'', the angle DCA by α, and the angle ECA by α'.



Then,

$$\text{tang. } \alpha = \frac{y'}{x'},$$

$$\text{tang. } \alpha' = \frac{y''}{x''}.$$

Therefore tang. $\alpha \times$ tang. $\alpha' = \frac{y'y''}{x'x''}$, which equals $-\frac{B^2}{A^2}$, because DD′ and EE′ are conjugate diameters, Prop. X., Schol.

Hence, by squaring each member of this equation, we have

$$A^4y'^2y''^2 = B^4x'^2x''^2. \qquad (1)$$

But because the points D and E are on the curve, we have

$$A^2y'^2 = A^2B^2 - B^2x'^2,$$

and

$$A^2y''^2 = A^2B^2 - B^2x''^2.$$

Therefore, by multiplication,

$$A^4y'^2y''^2 = A^4B^4 - A^2B^4x'^2 - A^2B^4x''^2 + B^4x'^2x''^2. \qquad (2)$$

Comparing equation (1) with equation (2), we see that

$$A^4B^4 - A^2B^4x'^2 - A^2B^4x''^2 = 0;$$

or, dividing by A^2B^4, we have

$$A^2 - x'^2 - x''^2 = 0,$$

or

$$A^2 = x'^2 + x''^2. \qquad (3)$$

In the same manner, we find that

$$B^2 = y'^2 + y''^2. \qquad (4)$$

Hence, by adding equations (3) and (4), we have

$$A^2 + B^2 = x'^2 + y'^2 + x''^2 + y''^2 = A'^2 + B'^2.$$

(89.) *Cor.* According to this Proposition, Equation (3),

$$x'^2 = A^2 - x''^2.$$

Also, from the equation of the ellipse, Art. 68,

$$A^2y''^2 = B^2(A^2 - x''^2).$$

Hence

$$x'^2 = \frac{A^2}{B^2}y''^2,$$

or

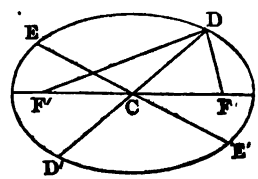
$$x' = \frac{A}{B}y''.$$

In the same manner, we find

$$y' = \frac{B}{A}x''.$$

PROPOSITION XIV.—THEOREM.

(90.) *If from the vertex of any diameter straight lines are drawn to the foci, their product is equal to the square of half the conjugate diameter.*



Represent the co-ordinates of the point D, referred to rectangular axes, by x', y'.

Then the square of the distance of D from the center of the ellipse is

$$A'^2 = x'^2 + y'^2.$$

But from the equation of the curve, Art. 68,

$$y'^2 = B^2 - \frac{B^2}{A^2}x'^2.$$

Therefore, by substitution,

$$A'^2 = B^2 + \frac{A^2 - B^2}{A^2}x'^2$$

$$= B^2 + e^2x'^2, \text{ Art. 69.}$$

But, by Prop. XIII., $A'^2 + B'^2 = A^2 + B^2$.

Therefore, $\qquad B'^2 = A^2 - e^2x'^2.$

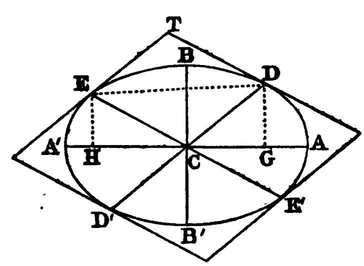
Also, by Prop. I., Cor. 7,

$$rr' = A^2 - e^2x'^2.$$

Hence $\qquad rr' = B'^2.$

PROPOSITION XV.—THEOREM.

(91.) *The parallelogram, formed by drawing tangents through the vertices of two conjugate diameters, is equal to the rectangle of the axes.*



Let DED′E′ be a parallelogram, formed by drawing tangents to the ellipse through the vertices of two conjugate diameters DD′, EE′; its area is equal to AA′×BB′.

Let the co-ordinates of D, referred to rectangular axes, be x', y', and those of E be x'', y''.

The triangle CDE is equal to the trapezium DEHG, diminished by the two triangles DCG, EHC. That is,

$$2.\text{CDE}=(x'+x'')(y'+y'')-x'y'-x''y'',$$
$$=x'y''+x''y',$$
$$=x'\frac{\text{B}x'}{\text{A}}+y'\frac{\text{A}y'}{\text{B}},\text{ by Prop. XIII., Cor.,}$$
$$=\frac{\text{B}^2x'^2+\text{A}^2y'^2}{\text{A.B}},\text{ reducing the fractions to a common denominator,}$$
$$=\frac{\text{A}^2\text{B}^2}{\text{A.B}}=\text{A.B, Art. 68.}$$

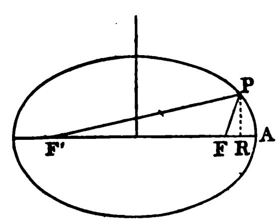
Therefore the parallelogram CETD is equal to A.B; and the parallelogram DED′E′ is equal to 4A.B or 2A×2B=AA′×BB′.

PROPOSITION XVI.—THEOREM.

(92.) *The polar equation of the ellipse, when the pole is at one of the foci, is*

$$r=\frac{\text{A}(1-e^2)}{1+e\cos. v},$$

where r is the radius vector, e is the eccentricity, and v is the angle which the radius vector makes with the major axis.



We have found the distance of any point of the ellipse from the focus, Prop. I., Cor. 7, to be

$$r=\text{FP}=\text{A}-ex,$$
$$r'=\text{F}'\text{P}=\text{A}+ex,$$

where the abscissa x is reckoned from the center. In order to transfer the origin from the center to the focus F, we must substitute for x,

$$x'+c;$$

or, putting Ae for c, Art. 69, we have

$$x=x'+\text{A}e.$$

If we represent the angle PFA by v, we shall have

$$x'=r\cos. v.$$

Whence $$x=r\cos. v+\text{A}e.$$

Therefore $$\text{FP}=r=\text{A}-er\cos. v-\text{A}e^2.$$

By transposition, $r(1+e\cos. v)=\text{A}-\text{A}e^2=\text{A}(1-e^2)$.

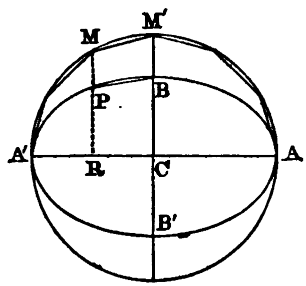
Whence $$r=\frac{\text{A}(1-e^2)}{1+e\cos. v}.$$

In the same manner, we find

$$r'=\frac{\text{A}(1-e^2)}{1-e\cos. v}.$$

PROPOSITION XVII.—THEOREM.

(93.) *The area of an ellipse is a mean proportional between the two circles described on its axes.*



Let AA′ be the major axis of an ellipse ABA′B′. On AA′, as a diameter, describe a circle; inscribe in the circle any regular polygon AM′MA′, and from the vertices M, M′, etc., of the polygon draw perpendiculars to AA′. Join the points B, P, etc., in which the perpendiculars intersect the ellipse, and there will be inscribed in the ellipse a polygon of an equal number of sides.

Let Y, Y′ be the ordinates of the points M, M′, and y, y' the ordinates of the points P, B, corresponding to the same abscissas x, x'.

The area of the trapezoid $M'MRC = \frac{Y+Y'}{2}(x-x')$.

The area of the trapezoid $BPRC = \frac{y+y'}{2}(x-x')$.

Whence $$\frac{BPRC}{M'MRC} = \frac{y+y'}{Y+Y'}.$$

But, by Prop. IV.,

$$y = \frac{B}{A}Y;\quad y' = \frac{B}{A}Y'.$$

Whence $$\frac{y+y'}{Y+Y'} = \frac{B}{A};$$

consequently, $$\frac{BPRC}{M'MRC} = \frac{B}{A}.$$

In the same manner it may be proved that each of the trapezoids composing the polygon inscribed in the ellipse, is to the corresponding trapezoid of the polygon inscribed in the circle, in the ratio of B to A; hence the entire polygon inscribed in the ellipse, is to the polygon inscribed in the circle, in the same ratio. Hence, if we represent the two polygons by p and P, we shall have

$$\frac{p}{P} = \frac{B}{A}.$$

Since this relation is true whatever be the number of sides of the polygons, it will be true when the number of the sides is indefinitely increased; that is, it is true for the ellipse and the circle, which are the limits of the surfaces of the polygons. Therefore, if we represent the surfaces of the ellipse and circle by s and S, we shall have

$$\frac{s}{S}=\frac{B}{A}, \text{ or } s=S\frac{B}{A}.$$

But the area of a circle whose radius is A, is represented by πA^2; hence the surface of the ellipse is

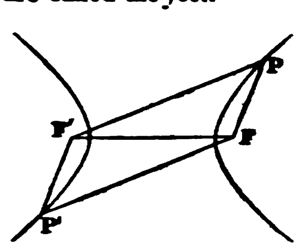
$$\pi A^2\frac{B}{A}=\pi AB,$$

which is a mean proportional between the two circles described on the axes. For the area of the circle described on the major axis is πA^2; and the area of that described on the minor axis is πB^2; and πAB is a mean proportional between them.

SECTION VII.

ON THE HYPERBOLA.

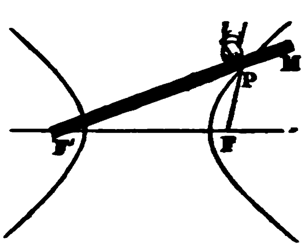
(94.) An *hyperbola* is a plane curve in which the difference of the distances of each point from two fixed points is equal to a given line. The two fixed points are called the *foci*.



Thus, if F and F' are two fixed points, and if the point P moves about F in such a manner that the *difference* of its distances from F and F' is always the same, the point P will describe an hyperbola, of which F and F' are the foci.

If the point P' moves about F' in such a manner that P'F−P'F' is always equal to PF'−PF, the point P' will describe a second hyperbola similar to the first. The two curves are called *opposite* hyperbolas.

(95.) This curve may be described by continuous motion as follows:



Let F and F' be any two fixed points. Take a ruler longer than the distance FF', and fasten one of its extremities at the point F'. Take a thread shorter than the ruler, and fasten one end of it at F, and the other to the end M of the ruler. Then move the ruler MPF' about the point F', while the thread is kept constantly stretched by a pencil pressed against the ruler; the curve described by the point of the pencil will be a portion of an hyperbola. For, in every position of the ruler, the difference of the lines PF, PF' will be the same, viz., the difference between the length of the ruler and the length of the string.

If the ruler be turned, and move on the other side of the point F, the other part of the same hyperbola may be described.

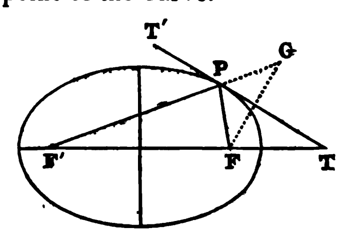
$$2A : 2Ae :: A+ex : F'N.$$

Hence $$F'N=e(A+ex).$$

But by Prop. VIII., Cor. 2, $e(A+ex)$ represents the distance from the focus F′ to the foot of the normal. Hence the line PN, which bisects the angle FPF′, is the normal.

(82.) *Cor.* 1. Since PN is perpendicular to TT′, and the angle FPN is equal to the angle F′PN, therefore the angle FPT is equal to the angle F′PT′; that is, *the radii vectores are equally inclined to the tangent.*

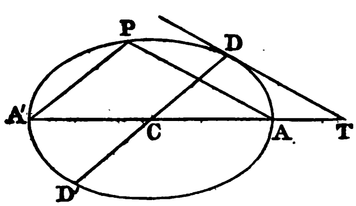
Cor. 2. This proposition affords a method of drawing a tangent line to an ellipse at a given point of the curve.



Let P be the given point; draw the radii vectores PF, PF′; produce PF′ to G, making PG equal to PF, and draw FG. Draw PT perpendicular to FG, and it will be the tangent required; for the angle FPT equals the angle GPT, which equals the vertical angle F′PT′.

Proposition X.—Theorem.

(83.) *If, through one extremity of the major axis, a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter which passes through the point of contact, and conversely.*



Let DT be a tangent to the ellipse, and let the chord AP be drawn parallel to it; then will A′P be parallel to the diameter DD′, which passes through the point of contact D.

Let x', y' designate the co-ordinates of D; the equation of the line CD will be, Art. 15,

$$y'=a'x';$$

whence $$a'=\frac{y'}{x'}.$$

But, by Prop. VII., Cor. 1, the tangent of the angle which the tangent line makes with the major axis, is

$$a=-\frac{B^2x'}{A^2y'}.$$

Multiplying together the values of a and a', we obtain

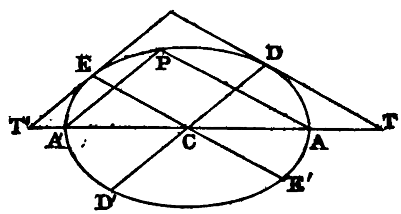
$$aa' = -\frac{B^2}{A^2},$$

which represents the product of the tangents of the angles DCT and DTC.

But by Prop. VI. the product of the tangents of the angles PAA′, PA′A is equal to $-\frac{B^2}{A^2}$.

Hence, if AP is parallel to DT, A′P will be parallel to CD, and conversely.

(84.) *Cor.* Let DD′ be any diameter of an ellipse, and DT the tangent drawn through its vertex, and let the chord AP be drawn parallel to DT; then, by this Proposition, the supplementary chord A′P is parallel to DD′. Let another tangent ET′ be drawn parallel to A′P, it will also be parallel to DD′. Let the diameter EE′ be drawn through the point of contact E; then, by this Proposition, A′P being parallel to T′E, AP (and, of course, DT) will be parallel to EE′. Each of the diameters DD′, EE′ is therefore parallel to a tangent drawn through the vertex of the other, and they are said to be *conjugate* to one another.



SCHOLIUM. Two diameters of an ellipse are said to be conjugate to one another, when each is parallel to a tangent line drawn through the vertex of the other.

If we designate by a and a' the tangents of the angles which two conjugate diameters make with the major axis, then we must have

$$aa' = -\frac{B^2}{A^2}.$$

PROPOSITION XI.—THEOREM.

(85.) *The equation of the ellipse, referred to its center and conjugate diameters, is*

$$A'^2y^2 + B'^2x^2 = A'^2B'^2;$$

where A′ and B′ are semi-conjugate diameters.

In this case the hyperbola is said to be *equilateral.*

Cor. 3. Since $$B^2=c^2-A^2,$$
and $$A^2+B^2=c^2 \text{ or } CF^2,$$
we see that *the square of the distance from the center to either focus is equal to the sum of the squares of the semi-axes.*

Cor. 4. According to the preceding Scholium,

$$y^2=\frac{B^2}{A^2}(x^2-A^2).$$

In order to determine the value of the parameter or double ordinate through the focus, make $x=c$ or CF; then

$$y^2=\frac{B^2}{A^2}(c^2-A^2).$$

But we have made $$B^2=c^2-A^2.$$
Hence $$y^2=\frac{B^2}{A^2}\times B^2,$$
or $$A : B :: B : y,$$
and $$2A : 2B :: 2B : 2y;$$
that is, *the parameter is a third proportional to the transverse and conjugate axes.*

Cor. 5. The quantity $\frac{c}{A}$, or the distance from the center to either focus, divided by the semi-transverse axis, is called the *eccentricity* of the hyperbola. If we represent the eccentricity by e, then

$$\frac{c}{A}=e, \text{ or } c=Ae.$$

But we have seen that

$$B^2=c^2-A^2.$$
Hence $$A^2+B^2=A^2e^2,$$
or $$\frac{B^2}{A^2}=e^2-1.$$

Making this substitution, the equation of the hyperbola becomes $$y^2=(e^2-1)(x^2-A^2).$$

Cor. 6. Equations (5) and (6) of the preceding Proposition are

$$r'=\ \ A+\frac{cx}{A},$$

$$r=-A+\frac{cx}{A}.$$

Substituting e for $\frac{c}{A}$, these equations become

$$r'=ex+A,$$
$$r=ex-A,$$

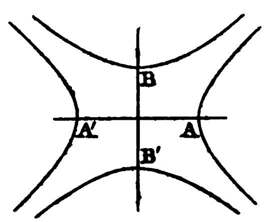
which equations represent the distance of any point of the hyperbola from either focus.

Multiplying these values together, we obtain

$$rr'=e^2x^2-A^2,$$

which is the value of the product of the focal distances.

SCHOLIUM 2. If on BB′, as a transverse axis, opposite hyperbolas are described having AA′ as their conjugate axis, these hyperbolas are said to be conjugate to the former.



The equation of the conjugate hyperbolas may be found from the equation

$$A^2y^2-B^2x^2=-A^2B^2,$$

by changing A into B and x into y. It then becomes

$$B^2x^2-A^2y^2=-A^2B^2,$$

which is the equation of the conjugate hyperbolas.

PROPOSITION II.—THEOREM.

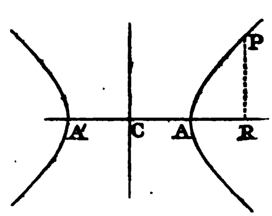
(99.) *The equation of the hyperbola, when the origin is at the vertex of the transverse axis, is*

$$y^2=\frac{B^2}{A^2}(x^2-2Ax),$$

where A and B represent the semi-axes, and x and y the general co-ordinates of the curve.

The equation of the hyperbola, when the origin is at the center, is, Art. 97,

$$A^2y^2-B^2x^2=-A^2B^2. \qquad (1)$$



If the origin is placed at A′, the ordinates will have the same value as when the origin was at the center, but the abscissas will be different.

If we represent the abscissas reckoned from A′ by x', then it is plain that we shall have

$$CR=A'R-A'C,$$

or $$x=x'-A.$$

Substituting this value of x in equation (1), we have

$$A^2y^2-B^2x'^2+2B^2Ax'=0,$$

which may be put under the form

$$y^2=\frac{B^2}{A^2}(x'^2-2Ax');$$

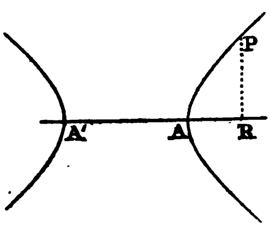
or, omitting the accents,

$$y^2=\frac{B^2}{A^2}(x^2-2Ax),$$

which is the equation of the hyperbola referred to the vertex A′ as the origin of co-ordinates.

Proposition III.—Theorem.

(100.) *The square of any ordinate is to the product of its distances from the vertices of the tranverse axis, as the square of the conjugate axis is to the square of the transverse axis.*



The equation of the hyperbola, referred to the vertex A′ as the origin of co-ordinates, is, Art. 99,

$$y^2=\frac{B^2}{A^2}(x-2A)x.$$

This equation may be resolved into the proportion

$$y^2 : (x-2A)x :: B^2 : A^2.$$

Now 2A represents the transverse axis AA′, and, since x represents A′R, $x-2A$ will represent AR; therefore, $(x-2A)x$ represents the product of the distances from the foot of the ordinate PR to the vertices of the transverse axis.

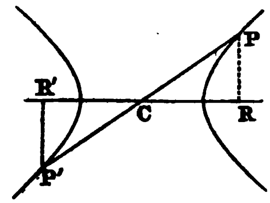
Cor. It is evident that the squares of any two ordinates are as the products of the parts into which they divide the transverse axis produced.

Proposition IV.—Theorem.

(101.) *Every diameter of an hyperbola is bisected at the center.*

Let PP′ be any diameter of an hyperbola. Let x', y' be the co-ordinates of the point P, and x'', y'' those of the point P′.

Then, from the equation of the curve, we shall have, Art. 98, Scholium 1,



$$y'^2=\frac{B^2}{A^2}(x'^2-A^2).$$

and $$y''^2=\frac{B^2}{A^2}(x''^2-A^2).$$

Whence $$\frac{y'^2}{y''^2}=\frac{x'^2-A^2}{x''^2-A^2}.$$

But from the similarity of the triangles PCR, P'CR', we have

$$\frac{y'}{y''}=\frac{x'}{x''}.$$

Whence $$\frac{x'^2}{x''^2}=\frac{x'^2-A^2}{x''^2-A^2}.$$

Clearing of fractions, we obtain

$$x'^2=x''^2.$$

Whence, also, $$y'^2=y''^2.$$

Consequently, $$x'^2+y'^2=x''^2+y''^2,$$

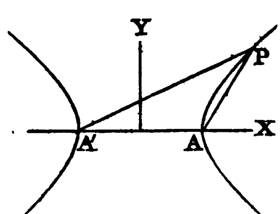
or $$CP^2=CP'^2;$$

that is, $$CP=CP'.$$

PROPOSITION V.—THEOREM.

(102.) *The product of the tangents of the two angles, formed by two lines drawn from the vertices of the transverse axis to meet in the curve, is equal to the square of the ratio of the semi-axes.*

The equation of the line AP passing through the point A, whose co-ordinates are $x'=A$, $y'=0$, Art. 18, is



$$y=a(x-A).$$

The equation of A'P passing through the point A', whose co-ordinates are $x'=-A$, $y'=0$, Art. 18, is

$$y=a'(x+A).$$

Since these lines are required to intersect on the curve, these two equations must not only exist at the same time, but also with the equation of the hyperbola. Multiplying the two equations together, we obtain

$$y^2=aa'(x^2-A^2).$$

And, since these lines intersect on the hyperbola, we must have, Art. 98,

$$y^2=\frac{B^2}{A^2}(x^2-A^2).$$

Comparing these two equations, we perceive that

$$aa'=\frac{B^2}{A^2}.$$

Cor. In the equilateral hyperbola A=B, and we have

$$aa'=1;$$

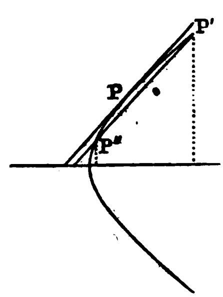
which shows that the angles formed by the supplementary chords, with the transverse axis on the same side, are together equal to a right angle, Art. 24.

PROPOSITION VI.—THEOREM.

(103.) *The equation of a straight line which touches an hyperbola is*

$$A^2yy'-B^2xx'=-A^2B^2,$$

where x and y are the general co-ordinates of the tangent line, x' and y' the co-ordinates of the point of contact.



Draw any line P′P″ cutting the hyperbola in the points P′, P″; if this line be moved toward P it will approach the tangent, and the secant will become a tangent when the points P′, P″ coincide.

Let x', y' be the co-ordinates of the point P′, and x'', y'' the co-ordinates of the point P″. The equation of the line P′P″, passing through these two points, will be, Art. 20,

$$y-y'=\frac{y'-y''}{x'-x''}(x-x'). \qquad (1)$$

Since the points P′, P″ are on the curve, we shall have, Art. 97,

$$A^2y'^2-B^2x'^2=-A^2B^2, \qquad (2)$$

$$A^2y''^2-B^2x''^2=-A^2B^2. \qquad (3)$$

Subtracting equation (3) from (2), we obtain

$$A^2(y'^2-y''^2)-B^2(x'^2-x''^2)=0,$$

or $$A^2(y'+y'')(y'-y'')-B^2(x'+x'')(x'-x'')=0.$$

Whence $$\frac{y'-y''}{x'-x''}=\frac{B^2}{A^2}\frac{(x'+x'')}{(y'+y'')}.$$

Substituting this value in equation (1), the equation of the secant line becomes

$$y-y'=\frac{B^2}{A^2}\frac{(x'+x'')}{(y'+y'')}(x-x'). \qquad (4)$$

The secant P′P″ will become a tangent when the points P′, P″ coincide, in which case

$$x'=x'', \text{ and } y'=y''.$$

Equation (4) in this case becomes

$$y-y'=\frac{B^2}{A^2}\frac{x'}{y'}(x-x'),$$

which is the equation of a tangent to the hyperbola at the point P. If we clear this equation of fractions, we obtain

$$A^2yy'-A^2y'^2=B^2xx'-B^2x'^2,$$

or $$A^2yy'-B^2xx'=-A^2B^2,$$

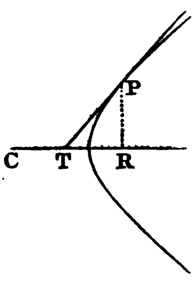
which is the most simple form of the equation of a tangent line.

(104.) *Cor.* 1. In the equation

$$y-y'=\frac{B^2x'}{A^2y'}(x-x'),$$

$\frac{B^2x'}{A^2y'}$ represents the trigonometrical tangent of the angle which the tangent line makes with the transverse axis.

Cor. 2. To find the point in which the tangent intersects the axis of abscissas, make $y=0$ in the equation of the tangent line, and we have



$$x=\frac{A^2}{x'},$$

which is equal to CT.

If from CR or x' we subtract CT, we shall have the subtangent

$$TR=x'-\frac{A^2}{x'}=\frac{x'^2-A^2}{x'}.$$

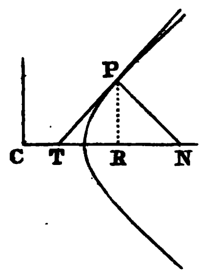
PROPOSITION VII.—THEOREM.

(105.) *The equation of a normal line to the hyperbola is*

$$y-y'=-\frac{A^2y'}{B^2x'}(x-x'),$$

where x and y are the general co-ordinates of the normal line, and x' and y' the co-ordinates of the point of intersection with the curve.

The equation of a straight line passing through the point whose co-ordinates are x', y', Art. 18, is



$$y-y'=a(x-x'); \qquad (1)$$

and, since the normal line is perpendicular to the tangent, we shall have, Art. 23,

$$a=\frac{1}{-a'}.$$

But we have found for the tangent line, Prop. VI., Cor. 1,

$$a'=\frac{B^2x'}{A^2y'}.$$

Hence $$a=-\frac{A^2y'}{B^2x'}.$$

Substituting this value in equation (1), we shall have for the equation of the normal line

$$y-y'=-\frac{A^2y'}{B^2x'}(x-x'). \qquad (2)$$

(106.) *Cor.* 1. To find the point in which the normal intersects the axis of abscissas, make $y=0$ in equation (2), and we have, after reduction,

$$CN=x=\frac{A^2+B^2}{A^2}x'.$$

If we subtract CR, which is represented by x', we shall have the subnormal

$$RN=\frac{A^2+B^2}{A^2}x'-x'=\frac{B^2x'}{A^2}.$$

Cor. 2. If we put $e^2=\frac{A^2+B^2}{A^2}$, Art. 98, Cor. 5, we shall have

$$CN=e^2x'.$$

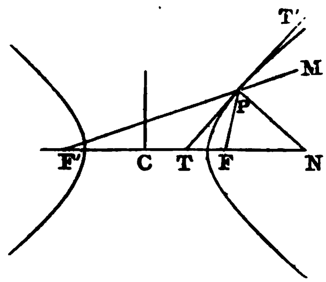
If to this we add F′C (see next figure), which equals c or Ae, Prop. I., Cor. 5, we have

$$F'N=Ae+e^2x'=e(A+ex'),$$

which is the distance from the focus to the foot of the normal.

PROPOSITION VIII.—THEOREM.

(107.) *A tangent to the hyperbola bisects the angle contained by lines drawn from the point of contact to the foci.*



Let PT be a tangent line to the hyperbola, and PF, PF' two lines drawn to the foci. Produce F'P to M, and draw PN bisecting the exterior angle FPM.

Then, by Geom., Prop. XVII., Schol., B. IV.,

$$F'P : FP :: F'N : FN;$$

or, by division,

$$F'P-FP :: F'F :: F'P : F'N. \qquad (1)$$

But $\quad F'P-FP=2A,$

$\quad F'F=2c=2Ae$, Prop. I., Cor. 5,

and $\quad F'P=A+ex$, Prop. I., Cor. 6.

Making these substitutions in proportion (1), we have

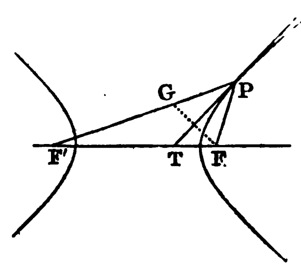
$$2A : 2Ae :: A+ex : F'N.$$

Hence $\quad F'N=e(A+ex).$

But, by Prop. VII., Cor. 2, $e(A+ex)$ represents the distance from the focus F' to the foot of the normal. Hence the line PN, which bisects the angle FPM, is the normal; that is, it is perpendicular to the tangent TT'. Now, since PN is perpendicular to TT', and the angle FPN is equal to the angle MPN, therefore the angle FPT is equal to MPT', or its vertical angle F'PT; that is, the tangent PT bisects the angle FPF'.

(108.) *Cor.* 1. The normal line PN bisects the exterior angle FPM, formed by two lines drawn to the foci.

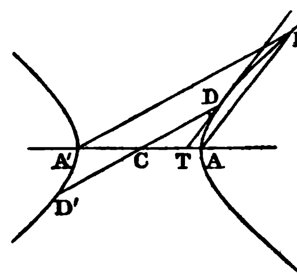
Cor. 2. This Proposition affords a method of drawing a tangent line to an hyperbola at a given point of the curve.



Let P be the given point; draw the radius vectors PF, PF'. On PF' take PG equal to PF, and draw FG. Draw PT perpendicular to FG, and it will be the tangent required, for it bisects the angle FPF'.

PROPOSITION IX.—THEOREM.

(109.) *If through one extremity of the transverse axis, a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter which passes through the point of contact, and conversely.*



Let DT be a tangent to the hyperbola, and let a chord AP be drawn parallel to it; then will A'P be parallel to the diameter DD' which passes through the point of contact D.

Let x', y' designate the co-ordinates of D; the equation of the line CD will be, Art. 15,

$$y'=a'x';$$

whence
$$a'=\frac{y'}{x'}.$$

But, by Prop. VI., Cor. 1, the tangent of the angle which the tangent line makes with the transverse axis, is

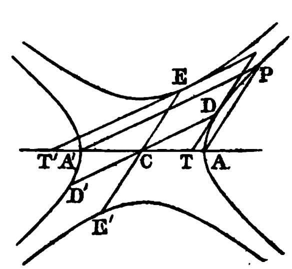
$$a=\frac{B^2x'}{A^2y'}.$$

Multiplying together the values of a and a', we obtain

$$aa'=\frac{B^2}{A^2},$$

which represents the product of the tangents of the angles DCT and DTC. But, by Prop. V., the product of the tangents of the angles PAA', PA'A is also equal to $\frac{B^2}{A^2}$.

Hence, if AP is parallel to DT, A'P will be parallel to CD, and conversely.



(110.) Let DD' be any diameter of an hyperbola, and DT the tangent drawn through its vertex, and let the chord AP be drawn parallel to DT; then, by this Proposition, the supplementary chord A'P is parallel to DD'. Let another tangent ET' be drawn to the conjugate hyperbola parallel to A'P, it will also

be parallel to DD′. Let the diameter EE′ be drawn through the point of contact E; then, by this Proposition, A′P being parallel to T′E, AP (and, of course, DT) will be parallel to EE′. Each of the diameters DD′, EE′ is, therefore, parallel to a tangent drawn through the vertex of the other, and they are said to be *conjugate* to one another.

SCHOLIUM. Two diameters of an hyperbola are said to be conjugate to one another, when each is parallel to a tangent line drawn through the vertex of the other.

If we designate by a and a' the tangents of the angles which two conjugate diameters make with the transverse axis, then we must have

$$aa'=\frac{B^2}{A^2}.$$

PROPOSITION X.—THEOREM.

(111.) *The equation of the hyperbola, referred to its center and conjugate diameters, is*

$$A'^2y^2-B'^2x^2=-A'^2B'^2,$$

where A′ and B′ are semi-conjugate diameters.

The equation of the hyperbola, referred to its center and axes, Art. 97, is

$$A^2y^2-B^2x^2=-A^2B^2.$$

In order to pass from rectangular to oblique co-ordinates, the origin remaining the same, we must substitute for x and y, in the equation of the curve, Art. 30, the values

$$x=x'\cos.\alpha+y'\cos.\alpha',$$
$$y=x'\sin.\alpha+y'\sin.\alpha'.$$

Squaring these values of x and y, and substituting in the equation of the hyperbola, we have

$$\begin{array}{r|l|l|l}
A^2\sin.^2\alpha' & y'^2+2A^2\sin.\alpha\sin.\alpha' & x'y'+A^2\sin.^2\alpha & x'^2=-A^2B^2, \\
-B^2\cos.^2\alpha' & \quad -2B^2\cos.\alpha\cos.\alpha' & \quad -B^2\cos.^2\alpha & \qquad\qquad (1)
\end{array}$$

which is the equation of the hyperbola when the oblique co-ordinates make any angles α, α' with the transverse axis.

But since the new axes are conjugate diameters, we must have, Art. 110,

$$aa'=\frac{B^2}{A^2},$$

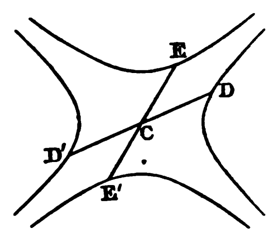
or $$\text{tang. } \alpha \text{ tang. } \alpha' = \frac{B^2}{A^2}.$$

Whence $$A^2 \text{ tang. } \alpha \text{ tang. } \alpha' - B^2 = 0.$$

Multiplying by $\cos. \alpha \cos. \alpha'$,
remembering that $\cos. \alpha \text{ tang. } \alpha = \sin. \alpha$,
we have $A^2 \sin. \alpha \sin. \alpha' - B^2 \cos. \alpha \cos. \alpha' = 0$.
Hence the term containing $x'y'$ in equation (1) disappears, and we have

$$(A^2 \sin.^2 \alpha' - B^2 \cos.^2 \alpha')y'^2 + (A^2 \sin.^2 \alpha - B^2 \cos.^2 \alpha)x'^2 = -A^2B^2, \quad (2)$$

which is the equation of the hyperbola referred to conjugate diameters.



If in this equation we make $y'=0$, we shall have

$$x'^2 = \frac{-A^2B^2}{A^2 \sin.^2 \alpha - B^2 \cos.^2 \alpha} = CD^2.$$

If we make $x'=0$, we shall have

$$y'^2 = \frac{-A^2B^2}{A^2 \sin.^2 \alpha' - B^2 \cos.^2 \alpha'} = CE^2.$$

If CD^2 is positive, CE^2 must be negative.

For when CD^2 is positive, the numerator of the above expression for its value being negative, its denominator must be negative; that is,

$$A^2 \sin.^2 \alpha < B^2 \cos.^2 \alpha,$$

or $$\text{tang. } \alpha^2 < \frac{B^2}{A^2}, \text{ Trig., Art. 28.}$$

But $$\text{tang.}^2 \alpha \text{ tang.}^2 \alpha' = \frac{B^4}{A^4}, \text{ Art. 110.}$$

Hence $$\text{tang.}^2 \alpha' > \frac{B^2}{A^2},$$

or $$A^2 \sin.^2 \alpha' > B^2 \cos.^2 \alpha';$$

that is, the denominator of the expression for CE^2 is positive; hence, since its numerator is negative, CE^2 must be negative.

If we represent CD^2 by A'^2, and CE^2 by $-B'^2$, equation (2) reduces to

$$\frac{y'^2}{-B'^2} + \frac{x'^2}{A'^2} = +1;$$

hence $$A'^2y'^2 - B'^2x'^2 = -A'^2B'^2;$$

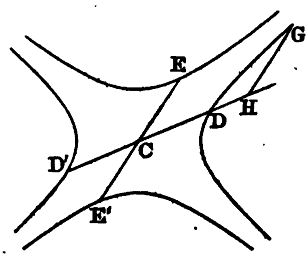
or, omitting the accents from x and y,

$$A'^2y^2-B'^2x^2=-A'^2B'^2,$$

which is the equation of the hyperbola referred to its center and conjugate diameters.

PROPOSITION XI.—THEOREM.

(112.) *The square of any diameter is to the square of its conjugate, as the rectangle of the segments from the vertices of the diameter to the foot of any ordinate, is to the square of that ordinate.*



The equation of the hyperbola, referred to conjugate diameters, Art. 111, is

$$A'^2y^2-B'^2x^2=-A'^2B'^2,$$

which may be put under the form

$$A'^2y^2=B'^2(x^2-A'^2).$$

This equation may be reduced to the proportion

$$A'^2 : B'^2 :: x^2-A'^2 : y^2,$$

or
$$(2A')^2 : (2B')^2 :: (x+A')(x-A') : y^2.$$

Now $2A'$ and $2B'$ represent the conjugate diameters DD′, EE′; and, since x represents CH, $x+A'$ will represent D′H, and $x-A'$ will represent DH; also, GH represents y^2; hence

$$DD'^2 : EE'^2 :: DH\times HD' : GH^2.$$

(113.) *Cor.* It is evident that the squares of any two ordinates to the same diameter, are as the rectangles of the corresponding segments from the vertices of the diameter to the foot of the ordinates.

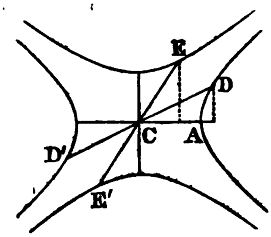
SCHOLIUM. The *parameter* of any diameter is a third proportional to the diameter and its conjugate. The parameter of the transverse axis is equal to $\frac{2B^2}{A}$, Art. 98, Cor. 4; and that of the conjugate axis is equal to $\frac{2A^2}{B}$.

PROPOSITION XII.—THEOREM.

(114.) *The difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.*

Let DD′, EE′ be any two conjugate diameters. Designate

the co-ordinates of D by x', y'; those of E by x'', y''; the angle DCA by α, and the angle ECA by α'. Then



$$\text{tang. } \alpha = \frac{y'}{x'},$$

$$\text{tang. } \alpha' = \frac{y''}{x''}.$$

Therefore, $\text{tang. } \alpha \times \text{tang. } \alpha' = \frac{y'y''}{x'x''}$,

which equals $\frac{B^2}{A^2}$, because DD' and EE' are conjugate diameters, Prop. IX., Schol.

Hence, by squaring each member of this equation, we have

$$A^4y'^2y''^2 = B^4x'^2x''^2. \qquad (1)$$

But because the point D is on the curve, we have, Art. 97,

$$A^2y'^2 = -A^2B^2 + B^2x'^2;$$

and because E is on the curve of the conjugate hyperbola, we have, Art. 98, Schol. 2,

$$A^2y''^2 = +A^2B^2 + B^2x''^2.$$

Therefore, by multiplication,

$$A^4y'^2y''^2 = -A^4B^4 + A^2B^4x'^2 - A^2B^4x''^2 + B^4x'^2x''^2. \qquad (2)$$

Comparing equation (1) with equation (2), we see that

$$-A^4B^4 + A^2B^4x'^2 - A^2B^4x''^2 = 0;$$

or, dividing by $-A^2B^4$, we have

$$A^2 - x'^2 + x''^2 = 0,$$

or $$A^2 = x'^2 - x''^2. \qquad (3)$$

In the same manner, we find that

$$B^2 = y''^2 - y'^2. \qquad (4)$$

Hence, by subtraction,

$$A^2 - B^2 = x'^2 + y'^2 - x''^2 - y''^2 = A'^2 - B'^2.$$

(115.) *Cor.* According to this Proposition, Equation (3),

$$x'^2 = A^2 + x''^2.$$

Also, from the equation of the hyperbola, Art. 98, Schol. 2,

$$A^2y''^2 = B^2(A^2 + x''^2).$$

Therefore $$x'^2 = \frac{A^2}{B^2}y''^2,$$

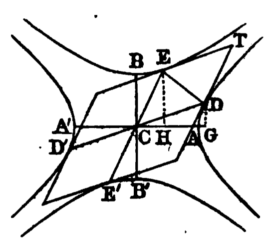
or $$x'=\frac{A}{B}y''.$$

In the same manner we find

$$y'=\frac{B}{A}x''.$$

PROPOSITION XIII.—THEOREM.

(116.) *The parallelogram formed by drawing tangents through the vertices of two conjugate diameters, is equal to the rectangle of the axes.*



Let DED′E′ be a parallelogram formed by drawing tangents to the hyperbola through the vertices of two conjugate diameters DD′, EE′; its area is equal to AA′×BB′.

Let the co-ordinates of D, referred to rectangular axes, be x', y'; and those of E be x'', y''.

The triangle CDE is equal to the trapezium DEHG, plus the triangle ECH, minus the triangle CDG; that is,

$$2.\text{CDE}=(x'-x'')(y'+y'')+x''y''-x'y',$$
$$=x'y''-x''y',$$
$$=x'\frac{Bx'}{A}-y'\frac{Ay'}{B}, \text{ by Prop. XII., Cor.,}$$
$$=\frac{B^2x'^2-A^2y'^2}{A.B}, \text{ reducing the fractions to a common denominator,}$$
$$=\frac{A^2B^2}{A.B}=A.B, \text{ Art. 97.}$$

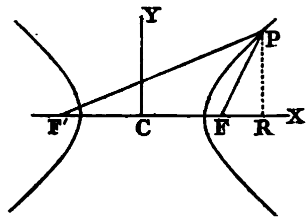
Therefore the parallelogram CETD is equal to A.B; and the parallelogram DED′E′ is equal to 4A.B or 2A×2B=AA′×BB′.

PROPOSITION XIV.—THEOREM.

(117.) *The polar equation of the hyperbola, when the pole is at the more remote focus, is*

$$r=\frac{A(1-e^2)}{1+e\cos. v},$$

where r is the radius vector, e is the eccentricity, and v is the angle which the radius vector makes with the transverse axis.



We have found the distance of any point of the hyperbola from the focus, Prop. I., Cor. 6, to be

$$r = FP = -A + ex,$$
$$r' = F'P = \quad A + ex,$$

where the abscissa x is reckoned from the center. In order to transfer the origin from the center to the focus F, we must substitute for x, $x'+c$; or, putting Ae for c, Art. 98, Cor. 5, we have

$$x = x' + Ae.$$

If we represent the angle PFX by v, we shall have

$$x' = r \cos. v.$$

Whence $\qquad x = r \cos. v + Ae.$

Therefore, $\qquad FP = r = -A + er \cos. v + Ae^2.$

By transposition,

$$r(1 - e \cos. v) = -A + Ae^2 = -A(1-e^2).$$

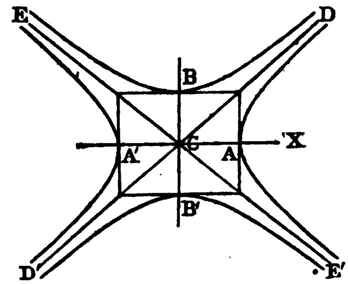
Whence $\qquad r = \dfrac{-A(1-e^2)}{1 - e \cos. v}.$

In the same manner, we find

$$r' = \frac{A(1-e^2)}{1 + e \cos. v}.$$

ON THE ASYMPTOTES OF THE HYPERBOLA.

(118.) If tangents to four conjugate hyperbolas be drawn through the vertices of the axes, the diagonals of the rectangle so formed, supposed to be indefinitely produced, are called *asymptotes* of the hyperbola.



Let AA′, BB′ be the axes of four conjugate hyperbolas, and through the vertices A, A′, B, B′, let tangents to the curve be drawn, and let DD′, EE′ be the diagonals of the rectangle thus formed; DD′, EE′ are called asymptotes to the curve.

If we represent the angle DCX by α, and the angle E′CX by α', then we shall have

$$\text{tang. } \alpha = \frac{B}{A},$$

$$\text{tang. } \alpha' = -\frac{B}{A}.$$

But, since $\text{tang. } \alpha = \frac{\sin. \alpha}{\cos. \alpha}$, Trig., Art. 28, we have

$$\frac{\sin.^2 \alpha}{\cos.^2 \alpha} = \frac{B^2}{A^2},$$

or $$\frac{\sin.^2 \alpha}{1 - \sin.^2 \alpha} = \frac{B^2}{A^2}.$$

Whence $$\sin.^2 \alpha = \frac{B^2}{A^2+B^2}.$$

In the same manner, we find

$$\cos.^2 \alpha = \frac{A^2}{A^2+B^2},$$

which equations furnish the value of the angle which the asymptotes form with the transverse axis.

PROPOSITION XV.—THEOREM.

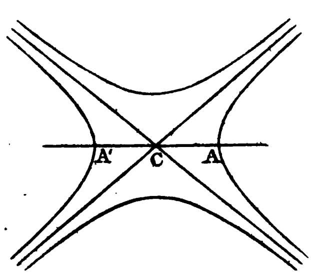
(119.) *The equation of the hyperbola, referred to its center and asymptotes, is*

$$xy = \frac{A^2+B^2}{4},$$

where A and B are the semi-axes, and x and y the co-ordinates of any point of the curve.

The equation of the hyperbola, referred to its center and axes, Art. 97, is

$$A^2y^2 - B^2x^2 = -A^2B^2. \quad (1)$$



The formulas for passing from rectangular to oblique co-ordinates, the origin remaining the same, Art. 30, are

$$x = x' \cos. \alpha + y' \cos. \alpha',$$
$$y = x' \sin. \alpha + y' \sin. \alpha'.$$

But, since $\alpha = -\alpha'$, these equations become

$$x = (x'+y') \cos. \alpha,$$
$$y = (x'-y') \sin. \alpha.$$

Substituting these values in equation (1), we have

$$A^2(x'-y')^2 \sin.^2 \alpha - B^2(x'+y')^2 \cos.^2 \alpha = -A^2B^2.$$

But $$\sin.^2 \alpha = \frac{B^2}{A^2+B^2}, \text{ Art. 118,}$$

and $$\cos.^2 \alpha = \frac{A^2}{A^2+B^2};$$

hence $$\frac{A^2B^2}{A^2+B^2}(x'-y')^2 - \frac{A^2B^2}{A^2+B^2}(x'+y')^2 = -A^2B^2;$$

that is, $$\frac{4A^2B^2}{A^2+B^2}x'y' = A^2B^2,$$

or $$x'y' = \frac{A^2+B^2}{4},$$

which is the equation of the hyperbola referred to its center and asymptotes.

(120.) *Cor. The curve of the hyperbola approaches nearer the asymptote the further it is produced, but, being extended ever so far, can never meet it.*

The equation of the hyperbola, referred to its asymptotes, Art. 119, is

$$xy = \frac{A^2+B^2}{4}.$$

Put M^2 for $\frac{A^2+B^2}{4}$, and we have

$$xy = M^2,$$

or $$y = \frac{M^2}{x};$$

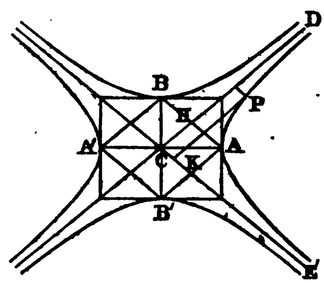
and, since M^2 is a constant quantity, y will vary inversely as x. Therefore y can not become zero until x becomes infinite; that is, the curve can not meet its asymptote except at an infinite distance from the center. The asymptotes are, therefore, considered as tangent to the curve at an infinite distance from the center.

PROPOSITION XVI.—THEOREM.

(121.) *If from any point of the hyperbola lines be drawn parallel to and terminating in the asymptotes, the parallelogram so formed will be equal to one eighth the rectangle described on the axes.*

Designate the co-ordinates of the point P referred to the

asymptotes by x', y', and the angle DCE′, included between the asymptotes, by β, we shall have, from the equation of the curve, Art. 119,



$$x'y' \sin. \beta = \frac{A^2+B^2}{4} \sin. \beta.$$

The first member of this equation represents the parallelogram PC contained by the co-ordinates of the point P of the curve.

The diagonal AB is equal to $\sqrt{A^2+B^2}$.

Hence $$AH \text{ or } CK = \frac{\sqrt{A^2+B^2}}{2}.$$

Also $$AK \text{ or } CH = \frac{\sqrt{A^2+B^2}}{2}.$$

Hence $$CH \times CK = \frac{A^2+B^2}{4}.$$

But the parallelogram

$$CHAK = CH \times CK \sin. \beta = \frac{A^2+B^2}{4} \sin. \beta.$$

Therefore the parallelogram PC is equal to the parallelogram HK, which is half the rectangle described on the semiaxes, or one eighth of the rectangle described on the axes.

PROPOSITION XVII.—THEOREM.

(122.) *The equation of a tangent line to an hyperbola, referred to its center and asymptotes, is*

$$y - y' = -\frac{y'}{x'}(x - x'),$$

where x', y' are the co-ordinates of the point of contact.

The equation of a secant line passing through the points x', y', x'', y'', Art. 20, is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'). \qquad (1)$$

Since the two given points are on the curve, we must have, Art. 120,

$$x'y' = M^2,$$
$$x''y'' = M^2.$$

Whence $$x'y' = x''y''$$

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Adding $x'y''$ to each member, we have

$$x'y'+x'y''=x''y''+x'y''.$$

Whence $$x'(y'-y'')=-y''(x'-x''),$$

or $$\frac{y'-y''}{x'-x''}=-\frac{y''}{x'}.$$

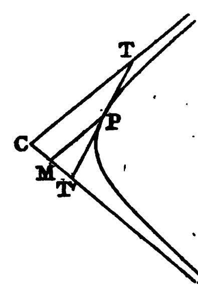
Hence, by substitution, equation (1) becomes

$$y-y'=-\frac{y''}{x'}(x-x'). \qquad (2)$$

If we suppose $x'=x''$, and $y'=y''$, the secant will become a tangent, and equation (2) will be

$$y-y'=-\frac{y'}{x'}(x-x'),$$

which is the equation of the tangent line.



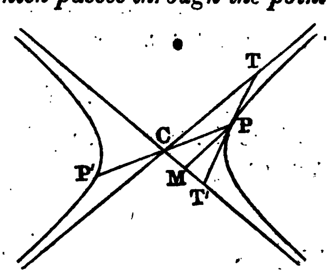
(123.) *Cor.* To find the point in which the tangent meets the axis of abscissas, make $y=0$ in the equation of the tangent line, and we have

$$x=2x';$$

that is, the abscissa CT′ of the point, where the tangent meets the asymptote CE, is double the abscissa CM of the point of tangency. Therefore CM=MT′; and, since the triangles TCT′, PMT′ are similar, the tangent TT′ is bisected in P, the point of contact; that is, *if a tangent line be drawn at any point of an hyperbola, the part included between the asymptotes is bisected at the point of tangency.*

PROPOSITION XVIII.—THEOREM.

(124.) *If a tangent line be drawn at any point of an hyperbola, the part included between the asymptotes is equal to the diameter which is conjugate to that which passes through the point of contact.*



Let TT′ be a line touching the hyperbola at P. Through P draw the diameter PP′, and designate the angle contained by the asymptotes by β.

Then, by Trigonometry, Art. 77, in the triangle CPM, we have

$$\cos. CMP = \frac{CM^2+MP^2-CP^2}{2CM \times MP},$$

or $$-\cos. \beta = \frac{x^2+y^2-CP^2}{2xy}.$$

Also, in the triangle PMT′,

$$\cos. PMT' = \frac{PM^2+MT'^2-PT'^2}{2PM \times MT'},$$

or $$\cos. \beta = \frac{x^2+y^2-PT'^2}{2xy}.$$

Whence we have

$$CP^2 = x^2+y^2+2xy \cos. \beta,$$
$$PT'^2 = x^2+y^2-2xy \cos. \beta.$$

Whence $$CP^2-PT'^2 = 4xy \cos. \beta.$$

But, since $\beta=2\alpha$, $\cos. \beta = \cos.^2 \alpha - \sin.^2 \alpha$, Trig., Art. 73.

Hence, from Art. 118, $\cos. \beta = \frac{A^2-B^2}{A^2+B^2}$.

Also, from the equation of the hyperbola, Art. 119,

$$xy = \frac{A^2+B^2}{4},$$

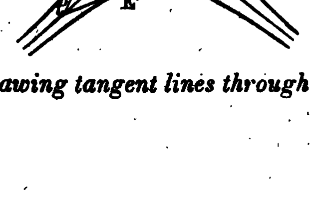
or $$4xy = A^2+B^2.$$

Therefore $$4xy \cos. \beta = A^2-B^2,$$

and $$CP^2-PT'^2 = A^2-B^2 = A'^2-B'^2 \text{ (Prop. XII.)}.$$

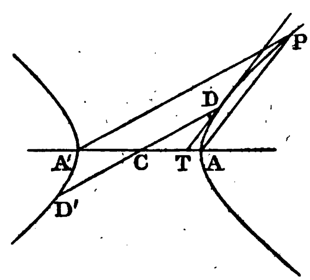
But CP is equal to A′; therefore PT′=B′; that is, the tangent TT′ is equal to the diameter which is conjugate to PP′.

(125.) *Cor.* The same is true of a tangent tt' drawn through the point P′ of the conjugate hyperbola. Therefore, if we join the points Tt, T′t', the figure Ttt'T′ will be a parallelogram whose sides are equal and parallel to 2A′, 2B′; that is, PP′, EE′. Hence *the asymptotes are the diagonals of all the parallelograms which can be formed by drawing tangent lines through the vertices of conjugate diameters.*



PROPOSITION IX.—THEOREM.

(109.) *If through one extremity of the transverse axis, a chord be drawn parallel to a tangent line to the curve, the supplementary chord will be parallel to the diameter which passes through the point of contact, and conversely.*



Let DT be a tangent to the hyperbola, and let a chord AP be drawn parallel to it; then will A′P be parallel to the diameter DD′ which passes through the point of contact D.

Let x', y' designate the co-ordinates of D; the equation of the line CD will be, Art. 15,

$$y'=a'x';$$

whence $$a'=\frac{y'}{x'}.$$

But, by Prop. VI., Cor. 1, the tangent of the angle which the tangent line makes with the transverse axis, is

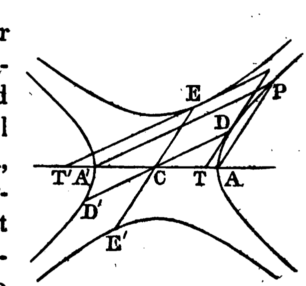
$$a=\frac{B^2x'}{A^2y'}.$$

Multiplying together the values of a and a', we obtain

$$aa'=\frac{B^2}{A^2},$$

which represents the product of the tangents of the angles DCT and DTC. But, by Prop. V., the product of the tangents of the angles PAA′, PA′A is also equal to $\frac{B^2}{A^2}$.

Hence, if AP is parallel to DT, A′P will be parallel to CD, and conversely.



(110.) Let DD′ be any diameter of an hyperbola, and DT the tangent drawn through its vertex, and let the chord AP be drawn parallel to DT; then, by this Proposition, the supplementary chord A′P is parallel to DD′. Let another tangent ET′ be drawn to the conjugate hyperbola parallel to A′P, it will also

But $$2 \sin. \alpha \cos. \alpha = \sin. 2\alpha,$$
and $$\cos.^2 \alpha - \sin.^2 \alpha = \cos. 2\alpha, \text{ Trig., Art. 73.}$$
Hence $$(A-C) \sin. 2\alpha + B \cos. 2\alpha = 0;$$
or, dividing by cos. 2α,

$$\text{tang. } 2\alpha = -\frac{B}{A-C}.$$

If, therefore, in equations (2), we give to the angle α such a value that the tangent of double that angle may be equal to $-\frac{B}{A-C}$, the term containing xy will disappear from the transformed equation. The new equation, therefore, becomes of the form

$$My^2+Nx^2+Ry+Sx+F=0. \qquad (3)$$

PROPOSITION II.—THEOREM.

(128.) *The terms containing the first power of the variables in the general equation of the second degree, can be made to disappear by changing the origin of the co-ordinates.*

In order to effect this transformation, substitute for x and y, in equation (3), the values

$$x=a+x',$$
$$y=b+y',$$

by which we pass from one system of axes to another system parallel to the first, Art. 28.

The result of this substitution is

$$My'^2+Nx'^2+\left.\begin{matrix}2Mb\\R\end{matrix}\right\} y'+\left.\begin{matrix}2Na\\S\end{matrix}\right\} x'+Mb^2+Na^2+Rb+Sa+F=0.$$

In order that the terms containing x' and y' may disappear, we must have

$$2Mb+R=0, \text{ or } b=-\frac{R}{2M},$$

and $$2Na+S=0, \text{ or } a=-\frac{S}{2N},$$

where a and b are the co-ordinates of the new origin.

If we employ these values of a and b, and substitute P for $-Mb^2-Na^2-Rb-Sa-F$, equation (3) reduces to

$$My^2+Nx^2=P,$$

an equation from which the terms containing the first power of the variables have been removed.

(129.) If one of the terms containing x^2 or y^2 was wanting from equation (3), this last result would be somewhat modified. If, for example, $N=0$, the value of a, given above, would reduce to $\frac{S}{0}$, or infinity. We can, however, in this case cause the term which is independent of the variables to disappear. For this purpose we must put

$$Mb^2+Rb+Sa+F=0$$

which gives $$a=-\frac{Mb^2+Rb+F}{S}.$$

With this value of a, and the value of $b=-\frac{R}{2M}$, equation (3) reduces to the form

$$My^2+Sx=0;$$

or, putting Q for $-\frac{S}{M}$, $$y^2=Qx.$$

Hence *every equation of the second degree between two variables may be reduced to one of the forms,*

$$My^2+Nx^2=P, \tag{4}$$

or $$y^2=Qx. \tag{5}$$

(130.) Equation (4) characterizes a circle, an ellipse, or an hyperbola.

First. Suppose M, N, and P are positive.

Put $$A^2=\frac{P}{N}, \text{ and } B^2=\frac{P}{M}.$$

By substituting these values in equation (4), we obtain

$$\frac{Py^2}{B^2}+\frac{Px^2}{A^2}=P,$$

or $$A^2y^2+B^2x^2=A^2B^2,$$

which is the equation of an ellipse, Art. 68.

If $M=N$, this equation characterizes a circle.

Secondly. If N and P are both negative, or the equation is of the form

$$My^2-Nx^2=-P,$$

put $$A^2=\frac{P}{N}, \text{ and } B^2=\frac{P}{M},$$

and we obtain, by substitution,

$$\frac{Py^2}{B^2}-\frac{Px^2}{A^2}=-P,$$

or $$A^2y^2-B^2x^2=-A^2B^2,$$

which is the equation of an hyperbola, Art. 97.

Thirdly. If N alone is negative, or the equation is of the form

$$My^2-Nx^2=P,$$

we shall obtain, by substitution, as before,

$$A^2y^2-B^2x^2=A^2B^2,$$

which characterizes the conjugate hyperbola, Art. 98, Schol. 2.

(131.) Equation (5) characterizes a parabola, since, by putting $Q=2p$, it becomes

$$y^2=2px, \text{ Art. 50.}$$

Hence *the only curves whose equations are of the second degree, are the circle, parabola, ellipse, and hyperbola.*

(132.) When the origin of co-ordinates is placed at the vertex of the transverse axis, the equation of the ellipse is, Art. 70,

$$y^2=\frac{B^2}{A^2}(2Ax-x^2).$$

The equation of the parabola for a similar position of the origin is, Art. 50,

$$y^2=2px;$$

and the equation of the hyperbola is, Art. 99,

$$y^2=\frac{B^2}{A^2}(2Ax+x^2).$$

The equation of the circle is

$$y^2=2Rx-x^2.$$

These equations may all be reduced to the form

$$y^2=mx+nx^2.$$

In the ellipse, $m=\frac{2B^2}{A}$, and $n=-\frac{B^2}{A^2}$.

In the parabola, $m=2p$, and $n=0$.

In the hyperbola, $m=\frac{2B^2}{A}$, and $n=\frac{B^2}{A^2}$.

In each case m represents the parameter of the curve, and n the square of the ratio of the semi-axes. In the ellipse, n is negative; in the hyperbola it is positive; and in the parabola it is zero.

(133.) Lines are divided into different orders, according to the degree of their equations.

A line of the first order has its equation of the form

$$Ay+Bx+C=0;$$

this class consists of the straight line only.

Lines of the second order have their equations of the form

$$Ay^2+Bxy+Cx^2+Dy+Ex+F=0.$$

This order comprehends four species, viz., the circle, ellipse, parabola, and hyperbola.

(134.) Lines of the third order have their equations of the form

$$Ay^3+By^2x+Cyx^2+Dx^3+Ey^2+Fyx+Gx^2+Hy+Kx+L=0.$$

Newton has shown that all lines of the third order are comprehended under some one of these four equations,

$$\begin{array}{ll} (1.) & xy^2+Ey=Ax^3+Bx^2+Cx+D, \\ (2.) & xy=Ax^3+Bx^2+Cx+D, \\ (3.) & y^2=Ax^3+Bx^2+Cx+D, \\ (4.) & y=Ax^3+Bx^2+Cx+D, \end{array}$$

in which A, B, C, D, E may be positive, negative, or evanescent, excepting those cases in which the equation would thus become one of an inferior order of curves.

He distinguished sixty-five different species of curves comprehended under the first equation; four new species were subsequently discovered by Sterling, and four more by De Gua.

The second equation comprehends only one species of curves, to which Newton has given the name of *Trident.*

The third equation includes five species, each possessing two parabolic branches; among these is the semi-cubical parabola.

The fourth equation comprehends only one species of curves, commonly called the cubical parabola.

There are, therefore, eighty different species of lines of the third order.

(135.) Lines of the fourth order have their equations of the form

$$\left.\begin{array}{r} Ay^4+By^3x+Cy^2x^2+Dyx^3+Ex^4 \\ +Fy^3+Gy^2x+Hyx^2+Kx^3 \\ +Ly^2+Myx+Nx^2 \\ +Py+Qx \\ +R \end{array}\right\}=0.$$

Lines of the fourth order are divided by Euler into 146 classes, and these comprise more than 5000 species.

As to the fifth and higher orders of lines, their number has precluded any attempt to arrange them in classes.

(136.) *A family* of curves is an assemblage of several curves of different kinds, all defined by the same equation of an indeterminate degree. Thus, every curve whose abscissas are proportional to any power of the ordinates is called a parabola. Hence the number of parabolas is indefinite. Of these some of the most remarkable have received specific names. The common parabola is sometimes called the quadratic parabola, and its equation is of the form . . . . $y^2=ax$.
The equation of the cubical parabola is . . . . . $y^3=ax$.
The equation of the biquadratical parabola is . . . $y^4=ax$.
etc. etc. etc.
The equation of the semi-cubical parabola is . . . $y^{\frac{3}{2}}=ax$.
The equation of the semi-biquadratical parabola is . $y^{\frac{4}{3}}=ax$.
etc. etc. etc.
All of these parabolas are included in the equation . $y^n=ax$.

SECTION II.

TRANSCENDENTAL CURVES.

(137.) Curves may be divided into two general classes, *algebraic* and *transcendental*.

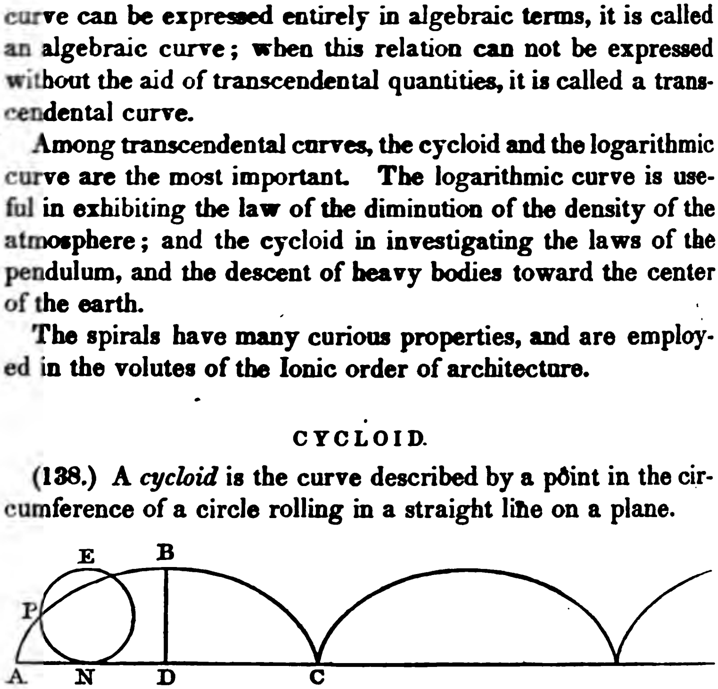
When the relation between the ordinate and abscissa of a curve can be expressed entirely in algebraic terms, it is called an algebraic curve; when this relation can not be expressed without the aid of transcendental quantities, it is called a transcendental curve.

Among transcendental curves, the cycloid and the logarithmic curve are the most important. The logarithmic curve is useful in exhibiting the law of the diminution of the density of the atmosphere; and the cycloid in investigating the laws of the pendulum, and the descent of heavy bodies toward the center of the earth.

The spirals have many curious properties, and are employed in the volutes of the Ionic order of architecture.

CYCLOID.

(138.) A *cycloid* is the curve described by a point in the circumference of a circle rolling in a straight line on a plane.



Thus, if the circle EPN be rolled along a straight line AC, any point P of the circumference will describe a curve which is called the cycloid. The circle EPN is called the *generating circle*, and P the *generating point.*

When the point P has arrived at C, having described the arc ABC, if it continue to move on, it will describe a second arc similar to the first, a third arc, and so on, indefinitely. As, however, in each revolution of the generating circle, an equal

curve is described, it is only necessary to examine the curve ABC described in one revolution of the generating circle.

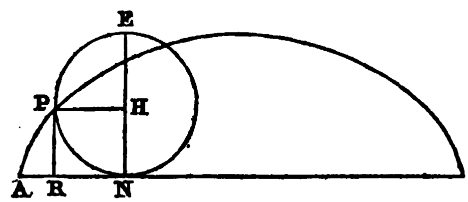
(139.) After the circle has made one revolution, every point of the circumference will have been in contact with AC, and the generating point will have arrived at C. The line AC will be equal to the circumference of the generating circle, and is called the *base of the cycloid.* The line BD, drawn perpendicular to the base at its middle point, is called the *axis of the cycloid,* and is equal to the diameter of the generating circle.

PROPOSITION I.—THEOREM.

(140.) *The equation of the cycloid is*

$$x = \text{arc whose versed sine is } y - \sqrt{2ry-y^2}$$

where r represents the radius of the generating circle.



Let us assume the point A as the origin of co-ordinates, and let us suppose that the generating point has described the arc AP. If N designate the point at which the generating circle touches the base, it is plain that the line AN will be equal to the arc PN. Through N draw the diameter EN, which will be perpendicular to the base. Through P draw PH parallel to the base, and PR perpendicular to it. Then PR will be equal to HN, which is the versed sine of the arc PN.

Let us put $EN=2r$, $AR=x$, and PR or $HN=y$; we shall then have, by Geom., Prop. XXII., Cor., B. IV.,

$$RN=PH=\sqrt{HN \times HE}=\sqrt{y(2r-y)}=\sqrt{2ry-y^2},$$

and $$AR=AN-RN=\text{arc } PN-PH.$$

Also, PN is the arc whose versed sine is HN or y.

Substituting the values of AR, AN, and RN, we have

$$x=\text{arc whose versed sine is } y-\sqrt{2ry-y^2},$$

which is the equation of the cycloid.

LOGARITHMIC CURVE.

(141.) The *logarithmic curve* takes its name from the property that, when referred to rectangular axes, any abscissa is

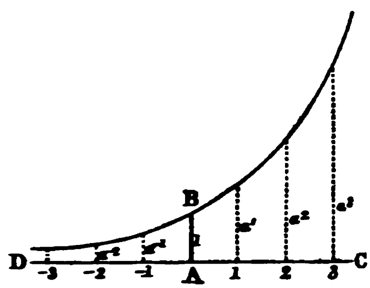
equal to the logarithm of the corresponding ordinate. The equation of the curve is, therefore,

$$x=\log.\ y.$$

If a represent the base of a system of logarithms, we shall have (Algebra, Art. 335),

$$a^x=y.$$

(142.) From this equation we can easily describe the curve by points. Let the line AB be taken for unity; and let AC be divided into portions, each equal to AB. Let a, the base of the system of logarithms, be taken equal to 1.6, and let a^2, a^3, etc., correspond in length with the different powers of a. Then the distances from A to 1, 2, 3, etc., will represent the logarithms of a, a^2, a^3, etc.

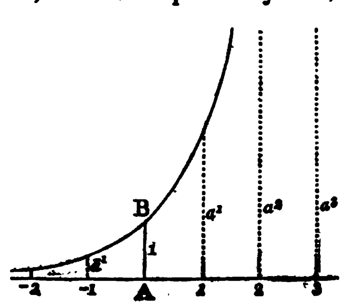


The logarithms of numbers less than a unit are *negative*, and these are represented by portions of the line AD to the left of the origin.

(143.) If the curve be continued ever so far to the left of A, it will never meet the axis of abscissas. The ordinates diminish more and more, but can never reduce to zero, while x is a finite quantity. When the ordinate becomes infinitely small, the abscissa becomes infinitely great and negative. This corresponds with Algebra (Art. 337), where it is shown that the logarithm of zero is infinite and negative.

(144.) We may construct the curve for any system of logarithms in a similar manner. Thus, for the Naperian system,

$$\begin{aligned} a &= 2.718, \\ a^2 &= 7.389, \\ a^3 &= 20.085, \\ a^{-1} &= 0.368, \\ a^{-2} &= 0.135. \end{aligned}$$



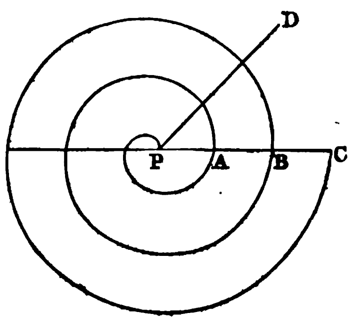
If we erect at the point A an ordinate equal to unity; at the point 1 an ordinate equal to 2.718; at the point 2 an ordi-

nate equal to 7.389, etc.; at the point -1 an ordinate equal to 0.368, etc., the curve passing through the extremities of these ordinates will be the logarithmic curve for the Naperian base.

SPIRALS.

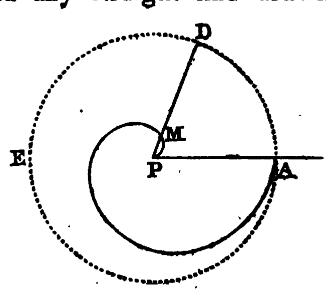
(145.) A *spiral* is a curve described by a point which moves along a right line in accordance with some prescribed law, the line having at the same time a uniform angular motion.

Thus, let PD be a straight line which revolves uniformly around the point P; and, at the same time, let a point move from P along the line PD, arriving successively at the points A, B, C, etc., it will trace out a curve called a spiral.



(146.) The fixed point P about which the right line moves, is called the *pole* of the spiral. The portion of the spiral generated while the line makes one revolution, is called a *spire;* and if the revolutions of the radius vector are continued, the generating point will describe an indefinite spiral, and any straight line drawn through the pole of the spiral, will intersect it in an infinite number of points.

With P as a center, and PA as a radius, describe the circumference ADE; the angular motion of the radius vector about the pole may be measured by the arcs of this circle estimated from A.



SPIRAL OF ARCHIMEDES.

(147.) While the line PD revolves uniformly about P, let the generating point move uniformly, also, along the line PD, it will describe the *spiral of Archimedes.*

PROPOSITION II.—THEOREM.

(148.) *The equation of the spiral of Archimedes is*

$$r=\frac{t}{2\pi},$$

where r represents the radius vector, and t the measuring arc estimated from A.

For, from the definition, the radius vectors are proportional to the measuring arcs estimated from A; that is,

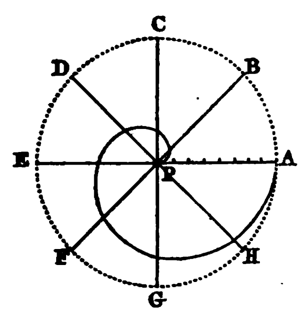
PM : PD :: arc AD : circ. ADE.

Designate the radius vector PM by r, PA by a, and the measuring arc estimated from A by t; then we shall have

$$r : a :: t : 2\pi a.$$

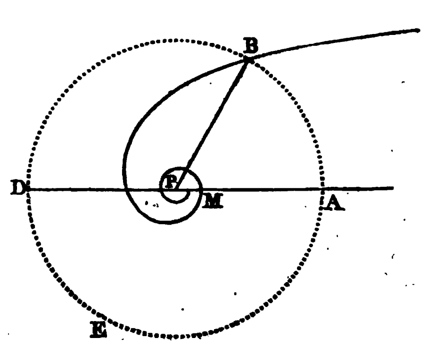
Whence $$r=\frac{at}{2\pi a}=\frac{t}{2\pi}.$$

(149.) This spiral may be constructed as follows: divide a circumference into any number of equal parts, as, for example, eight; and the radius AP into the same number of equal parts. On the radius PB lay off one of these parts; on PC two; on PD three, etc. The curve passing through these points will be the spiral of Archimedes, for the radius vectors are proportional to the arcs AB, BC, etc., of the measuring circle.



HYPERBOLIC SPIRAL.

(150.) While the line PB revolves uniformly about P, let the generating point move along the line PB in such a manner, that the radius vectors shall be inversely proportional to the corresponding arcs, it will describe the *hyperbolic spiral.*



PROPOSITION III.—THEOREM.

(151.) *The equation of the hyperbolic spiral is*

$$r=\frac{a}{t},$$

where r represents the radius vector, t the measuring arc, and a is a constant quantity.

For, from the definition,

PB : PM : : circ. ABDE : arc AB.

Let us designate the radius vector by r, and the measuring arc estimated from A by t, calling PM unity, we shall have

$$r : 1 : : 2\pi : t.$$

Whence $$r=\frac{2\pi}{t};$$

or, representing the constant 2π by a, we have

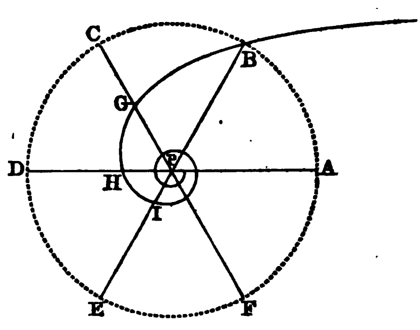
$$r=\frac{a}{t}, \text{ or } at^{-1}.$$

(152.) SCHOLIUM. The two preceding spirals are included in the general equation

$$r=at^n,$$

where n may be either positive or negative.

(153.) The hyperbolic spiral may be constructed as follows: Describe a circle ACDF, and divide its circumference into any number of equal parts, AB, BC, CD, etc. Then take PG equal to one half of PB, PH equal to one third of PB, PI equal to one fourth of PB, etc., the curve passing through the points B, G, H, I, etc., will be a hyperbolic spiral, because the radius vectors are inversely proportional to the corresponding arcs estimated from A.



LOGARITHMIC SPIRAL.

(154.) While the line PA revolves uniformly about P, let the generating point move along PA in such a manner that

the logarithm of the radius vector may be proportional to the measuring arcs, it will describe the *logarithmic spiral.*

PROPOSITION IV.—THEOREM.

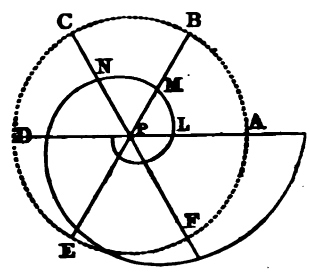
(155.) *The equation of the logarithmic spiral is*

$$t = \log. r,$$

where r represents the radius vector, and t the measuring arc.

For this equation is but an expression of the definition.

(156.) The logarithmic spiral may be constructed as follows: Divide the arc of a circle ACE into any number of equal parts, AB, BC, CD, etc., and upon the radii drawn to the points of division, take PL, PM, PN, etc., in geometrical progression. The curve passing through the points L, M, N, etc., will be the logarithmic spiral; for it is evident that AB, AC, etc., being in arithmetical progression, are as the logarithms of PL, PM, etc., which are in geometrical progression.



DIFFERENTIAL CALCULUS.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES—DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

ARTICLE (157.) IN the Differential Calculus, as in Analytical Geometry, there are two classes of quantities to be considered, viz., *variables* and *constants.*

Variable quantities are generally represented by the last letters of the alphabet, x, y, z, etc., and any values may be assigned to them which will satisfy the equations into which they enter.

Constant quantities are generally represented by the first letters of the alphabet, a, b, c, etc., and these always retain the same values throughout the same investigation.

Thus, in the equation of a straight line,

$$y=ax+b,$$

the quantities a and b have but one value for the same line, while x and y vary in value for every point of the line.

(158.) One variable is said to be a *function* of another variable, when the first is equal to a certain algebraic expression containing the second. Thus, in the equation of a straight line,

$$y=ax+b,$$

y is a function of x.

So, also, in the equation of a circle,

$$y=\sqrt{R^2-x^2};$$

and in the equation of the ellipse,

$$y=\frac{B}{A}\sqrt{2Ax-x^2}.$$

(159.) When we wish merely to denote that y is *dependent* upon x for its value, without giving the particular expression which shows the value of x, we employ the notation

$$y=F(x), \text{ or } y=f(x),$$

or

$$x=F(y), \text{ or } x=f(y),$$

which expressions are read, y is a function of x, or x is a function of y.

To denote a function containing two independent variables, as x and y, we inclose the variables in a parenthesis, and place the sign of function before them. Thus, the equation

$$u=ay+bx^2$$

may be expressed generally by

$$u=f(x, y),$$

which is read, u is a function of x and y, and simply shows that u is dependent for its value upon both x and y.

(160.) An *explicit* or expressed function is one in which the value of the function is directly expressed in terms of the variable and constants, as in the equation

$$y=ax^2+b.$$

An *implicit* or implied function is one in which the value of the function is not directly expressed in terms of the variable and constants, as in the equation

$$y^3-3ayx+x^3=0,$$

where the form of the function that y is of x can be ascertained only by solving the equation.

(161.) An *increasing* function is one which is increased when the variable is increased, or decreased when the variable is decreased. Thus, in the equation of a straight line,

$$y=ax+b,$$

if the value of x is increased, the value of y will also increase; or, if x is diminished, the value of y will diminish.

A *decreasing* function is one which is decreased when the variable is increased, and increased when the variable is decreased. Thus, in the equation of the circle,

$$y=\sqrt{R^2-x^2},$$

the value of y increases when x is diminished, and decreases when x is increased.

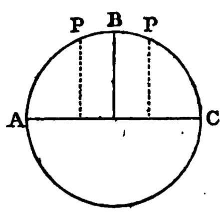
In the equation $y=\sqrt{R^2-x^2}$, x is called the *independent variable*, and y the *dependent variable*, because arbitrary values are supposed to be assigned to x, and the corresponding values of y are deduced from the equation.

(162.) The *limit* of a variable quantity is that value which it continually approaches, so as, at last, to differ from it by less than any assignable quantity.

Thus, if we have a regular polygon inscribed in a circle, and if we inscribe another polygon having twice the number of sides, the area of the second will come nearer to the area of the circle than that of the first. By continuing to double the number of sides, the area of the polygon will approach nearer and nearer to that of the circle, and may be made to differ from it by a quantity less than any finite quantity. Hence the circle is said to be the limit of all its inscribed polygons.

So, also, in the equation of a circle,

$$x^2+y^2=R^2,$$



the value of y increases as the point P advances from A to B, at which point it becomes equal to the radius of the circle. As the point P advances from B to C, the value of y diminishes until at C it is reduced to zero. The radius of the circle is, therefore, the limit which the value of y can never exceed. So, also, in the same equation, the radius of the circle is the limit which the value of x can never exceed.

If we convert $\frac{1}{9}$ into a decimal fraction, it becomes

$$.1111, \text{ etc.},$$

or $$\tfrac{1}{10}+\tfrac{1}{100}+\tfrac{1}{1000}+\tfrac{1}{10000}+, \text{ etc.}$$

Hence the sum of the terms of this series approaches to the value of $\frac{1}{9}$, but can never equal it while the number of terms is finite. The limit of the sum of the terms of this series is therefore $\frac{1}{9}$.

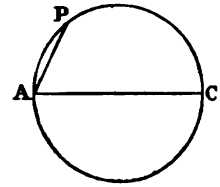
So, also, the sum of the series,

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}, \text{ etc.},$$

approaches nearer to 2, the greater the number of terms we employ; and, by taking a sufficient number of terms, the sum of the series may be made to differ from 2 by less than any quantity we may please to assign. The limit of the sum of the terms of this series is therefore 2.

(163.) When two magnitudes decrease simultaneously, they may approach continually toward a ratio of equality, or toward some other definite ratio. Thus, let a point P be sup-

posed to move on the circumference of a circle toward a fixed point A. The arc AP will diminish at the same time with the chord AP, and, by bringing the point P sufficiently near to A, we may obtain an arc and its chord, each of which shall be smaller than any given line, and the arc and the chord continually approach to a ratio of equality.



But the ratio of two magnitudes does not necessarily approach to equality, because the magnitudes are indefinitely diminished. Thus, take the two series,

$$1,\ \tfrac{1}{3},\ \tfrac{1}{6},\ \tfrac{1}{10},\ \tfrac{1}{15},\ \tfrac{1}{21},\ \tfrac{1}{28},\ \text{etc.},$$
$$1,\ \tfrac{1}{4},\ \tfrac{1}{9},\ \tfrac{1}{16},\ \tfrac{1}{25},\ \tfrac{1}{36},\ \tfrac{1}{49},\ \text{etc.}$$

The ratio of the corresponding terms is,

$$1,\ \tfrac{4}{3},\ \tfrac{9}{6},\ \tfrac{16}{10},\ \tfrac{25}{15},\ \tfrac{36}{21},\ \tfrac{49}{28},\ \text{etc.}$$

The ratio here increases at every step, but not without limit. However far the two series may be continued, the ratio of the corresponding terms is never so great as 2, though it may be made to differ from 2 by less than any assignable quantity. The limit of the ratio of the corresponding terms of the two series is therefore 2.

(164.) If a variable quantity increase uniformly, then other quantities, depending on this and constant quantities, may either vary uniformly, or according to any law whatever.

Thus, in the equation of a straight line,

$$y=2x+3,$$

if we make	$x=1$,	we find	$y=5$,
	$x=2$,	"	$y=7$,
	$x=3$,	"	$y=9$,
	etc.		etc.;

that is, when x increases uniformly, y increases uniformly.

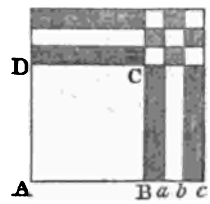
Again, take the equation of the parabola,

$$y=\sqrt{4x},$$

if we make	$x=1$,	we find	$y=2.000$,
	$x=2$,	"	$y=2.828$,
	$x=3$,	"	$y=3.464$,
	$x=4$,	"	$y=4.000$,
	etc.		etc.,

where, although x increases uniformly, y does not increase uniformly.

(165.) If the side of a square increases uniformly, the area does not increase uniformly. Thus, let AB be the side of a square, and let it increase uniformly by the additions Ba, ab, bc, etc., and let squares be described on these lines, as in the annexed figure; then it is obvious that the square on the side Aa exceeds that described on the side AB, by twice the rectangle AB×Ba, together with the square on Ba. The square described on Ab has received a further increment of two equal rectangles, together with three times the square on Ba; the square on AC has received a further increment of two equal rectangles and five times the square on Ba. Hence, when the side of the square varies uniformly, the area does not vary uniformly.



Thus, suppose the side of a square is equal to 10 inches, and let it increase uniformly one inch per minute, so as to become successively 11, 12, etc., inches.

While the side increases from 10 to 11 inches, the area increases from 100 to 121 inches=21 inches.

While the side increases from 11 to 12 inches, the area increases from 121 to 144 inches=23 inches.

While the side increases from 12 to 13 inches, the area increases from 144 to 169 inches=25 inches.

etc., etc., etc.

Hence the rate of increase of the area depends upon the length of the side. When the side is 11 inches, the area is increasing more rapidly than when the side was 10 inches.

(166.) There is an important distinction between the *absolute increase* of a variable quantity, and its *rate of increase.* By the *rate* of increase at any instant we understand what *would have been* the absolute increase if this increase had been uniform. Thus, while the side of a square increases from 11 to 12 inches in one minute, the area increases from 121 to 144 inches. The absolute increase of the area is 23 inches; but the *rate* of increase of the area when the side was 11 inches was such as would have given an increase of *less* than 23 inches per minute; and when the side was 12 inches the

rate of increase was such as would have given an increase of more than 23 inches per minute.

While, therefore, the rate of increase of the side of a square is uniform, the rate of increase of its area is continually changing.

The object of the Differential Calculus is to determine the ratio between the rate of variation of the independent variable, and that of the function into which it enters.

PROPOSITION I.—THEOREM.

(167.) *The rate of variation of the side of a square is to that of its area, in the ratio of unity to twice the side of the square.*

If the side of a square be represented by x, its area will be represented by x^2. When the side of the square is increased by h and becomes $x+h$, the area will become $(x+h)^2$, which is equal to

$$x^2+2xh+h^2.$$

While the side has increased by h, the area has increased by $2xh+h^2$. If, then, we employ h to denote the rate at which x increases, $2xh+h^2$ would have denoted the rate at which the area increased had that rate been uniform; in which case we should have had the following proportion:

rate of increase of the side : *rate of increase of the area* :: $h : 2xh+h^2$,
or as $\qquad 1 : 2x+h.$

But since the area of the square increases each instant more and more rapidly, the quantity $2x+h$ is greater than the increment which would have resulted had the rate at which the square was increasing when its side became x continued uniform; and the smaller h is supposed to be, the nearer does the increment $2x+h$ approach to that which would have resulted had the rate at which the square was increasing when its side became x continued uniform. When h is equal to zero, this ratio becomes that of

$$1 \text{ to } 2x,$$

which is, therefore, the ratio of the rate of increase of the side to that of the area of the square, when the side is equal to x.

(168.) ILLUSTRATION. If the side of a square be 10 feet, its area will be 100 feet. If the side be increased to 11 feet, its

area will become 121 feet; the area has increased 21 feet; and the ratio of the increment of the side to that of the area is as 1 to 21.

When the length of the side was 10.1 feet, its area was 102.01; the area had increased 2.01 feet; and the ratio of the increment of the side to that of the area, was as 0.1 to 2.01, or 1 to 20.1.

When the length of the side was 10.01 feet, its area was 100.2001; and the ratio of the increment of the side to that of the area, was as 1 to 20.01.

When the length of the side was 10.001 feet, the ratio was as 1 to 20.001.

Hence we see that the smaller is the increment of the side of the square, the nearer does the ratio of the increments of the side and area approach to the ratio of 1 to 20. This, therefore, was the ratio of the rates of increase at the instant the side was equal to 10 feet; and this ratio is that of one to twice ten, or twice the side of the square.

We have here another illustration of the principle of Art. 163, that two magnitudes which decrease simultaneously may continually approach toward some finite ratio. However small we suppose the increment of the side of the square or the increment of the area to become, the ratio of the two increments continually approaches to that of 1 to $2x$.

PROPOSITION II.—THEOREM.

(169.) *The rate of variation of the edge of a cube is to that of its solidity, in the ratio of unity to three times the square of the edge.*

If the edge of a cube be represented by x, and its solidity by u, then

$$u=x^3.$$

If the edge of the cube be increased by h so as to become $x+h$, and the corresponding solidity be represented by u', then we shall have

$$u'=(x+h)^3=x^3+3x^2h+3xh^2+h^3.$$

The increment of the cube is

$$u'-u=3x^2h+3xh^2+h^3.$$

Hence, if the solidity of the cube had increased uniformly when

the edge increased uniformly, we should have had the proportion,

rate of increase of the edge : rate of increase of the solidity

$$:: h : 3x^2h+3xh^2+h^3,$$

or as

$$1 : 3x^2 +3xh +h^2.$$

But since the solidity at each instant increases more and more rapidly, the increment $3x^2+3xh+h^2$ is greater than that which would have resulted had the rate of increase when its edge became x continued uniform. Now the smaller h becomes, the nearer does the increment $3x^2+3xh+h^2$ approach to that which would have resulted had the rate at which the cube was increasing when its edge became x continued uniform. When h is equal to zero, this ratio becomes that of

$$1 \text{ to } 3x^2,$$

which is, therefore, the ratio of the rate of increase of the edge to that of the solidity, when the edge is equal to x.

(170.) The rate of variation of a function or of any variable quantity is called its *differential*, and is denoted by the letter d placed before it. Thus, if

$$u=x^3,$$

then

$$dx : du :: 1 : 3x^2.$$

The expressions dx, du are read differential of x, differential of u, and denote the rates of variation of x and u.

If we multiply together the extremes and the means of the preceding proportion, we have

$$du=3x^2dx,$$

which signifies that the rate of increase of the function u is $3x^2$ times that of the variable x.

If we divide each member of the last equation by dx, we have

$$\frac{du}{dx}=3x^2,$$

which expresses the ratio of the rate of variation of the function to that of the independent variable, and is called the *differential coefficient* of u regarded as a function of x.

(171.) ILLUSTRATION. If the edge of a cube be 10 feet, its solidity will be 1000 feet. If the edge be increased to 11 feet, its solidity will be 1331 feet; the solidity has increased 331

feet; and the ratio of the increment of the edge to that of the solidity is as 1 to 331.

When the length of the edge was 10.1 feet, its solidity was 1030.301; the solidity had increased 30.301 feet; and the ratio of the increment of the edge to that of the solidity was as 0.1 to 30.301, or 1 to 303.01.

When the length of the edge was 10.01 feet, its solidity was 1003.003001; and the ratio of the increment of the edge to that of the solidity was as 1 to 300.3001.

When the length of the edge was 10.001 feet, the ratio was as 1 to 300.030001.

Hence we see that the smaller is the increment of the edge of the cube, the nearer does the ratio of the increments of the edge and solidity approach to the ratio of 1 to 300. This, therefore, was the ratio of the rates of increase at the instant the edge was equal to 10 feet; and this ratio is that of one to three times the square of ten.

(172.) It will be seen from these examples that, in order to discover the rate of variation of a function, we ascribe a small increment to the independent variable, and learn the corresponding increment of the function. We then observe toward what limit the ratio of these increments approaches, as the increment of the variable is diminished, which limit can only be attained when the increment of the variable is supposed to become zero. This limit expresses the ratio of the rates of variation of the function and the independent variable x, at the instant when the variable was equal to x.

But because, in order to find the value of the differential coefficient, we make h equal to zero, it must not be inferred that dx and du are therefore equal to zero. du denotes the *rate* of variation of the function u, and dx the *rate* of variation of the variable x; and since only their ratio is determined, either of them may have any value whatever. dx may, therefore, be supposed to have a very small or a very large value at pleasure.

PROPOSITION III.—THEOREM.

(173.) *The differential coefficient of the function*

$$u=x^4$$

is $4x^3$.

If we suppose x to be increased by any quantity h, and des-

ignate by u' the new value of the function under this supposition, we shall have

$$u'=(x+h)^4;$$

or, expanding the second member of the equation, we have

$$u'=x^4+4x^3h+6x^2h^2+4xh^3+h^4.$$

If we subtract from this the original equation, we obtain

$$u'-u=4x^3h+6x^2h^2+4xh^3+h^4.$$

Hence we see that if the variable x is increased by h, the function u will be increased by

$$4x^3h+6x^2h^2+4xh^3+h^4.$$

If both members of the last equation be divided by h, we shall have

$$\frac{u'-u}{h}=4x^3+6x^2h+4xh^2+h^3,$$

which expresses the ratio of the increment of the function u to that of the variable x. The first term $4x^3$ of this ratio is independent of h, so that, however we vary the value of h, this first term will remain unchanged, but the subsequent terms are dependent on h.

If we suppose h to diminish continually, the value of this ratio will approach to that of $4x^3$, to which it will become equal when h equals zero. This, therefore, is the ratio of the rate of increase of the independent variable to that of the function, at the instant the variable was equal to x. Hence, Art. 172,

$$\frac{du}{dx}=4x^3.$$

(174.) The method here exemplified is applicable to the determination of the differential coefficient of any function of a single variable, and is expressed in the following

RULE.

Give to the variable any arbitrary increment h, *and find the corresponding value of the function; from which subtract its primitive value. Divide the remainder by the increment* h, *and find the limit of this ratio, by making the increment equal to zero; the result will be the differential coefficient.*

Ex. 1. If x increase uniformly at the rate of 2 inches per

second, at what rate does the value of the expression $2x^2$ increase when x equals 6 inches?

Ans. 48 inches per second.

Ex. 2. If x increase uniformly at the rate of 3 inches per second, at what rate does the value of the expression $4x^3$ increase when x equals 10 inches?

Ans.

Ex. 3. If x increase uniformly at the rate of 5 inches per second, at what rate does the value of the expression $2x^4$ increase when x equals 4 inches?

Ans.

PROPOSITION IV.—THEOREM.

(175.) *To obtain the differential of any power of a variable, we must diminish the exponent of the power by unity, and then multiply by the primitive exponent, and by the differential of the variable.*

To prove this proposition, let us take the function

$$u=x^n,$$

and suppose x to become $x+h$, then

$$u'=(x+h)^n.$$

Developing the second member of this equation by the Binomial theorem, we have

$$u'=x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\text{, etc.}$$

Subtracting from this the original equation, we have

$$u'-u=nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\text{, etc.}$$

Dividing both members by h, we have

$$\frac{u'-u}{h}=nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\text{, etc.}$$

which expresses the ratio of the increment of the function to that of the variable.

If now we make h equal to zero, Art. 174, the second term of the second member of this equation reduces to zero, and also all the subsequent terms of the development, since they contain powers of h. Hence

$$\frac{du}{dx}=nx^{n-1},$$

or $$du=nx^{n-1}dx,$$

which conforms to the proposition above enunciated.

PROPOSITION V.—THEOREM.

(176.) *The differential of the product of a variable quantity by a constant, is equal to the constant multiplied by the differential of the variable.*

Suppose we have the function

$$u=ax^4.$$

When x becomes $x+h$, we have

$$u'=ax^4+4ax^3h+6ax^2h^2+, \text{ etc.}$$

Also, $$u'-u=4ax^3h+6ax^2h^2+, \text{ etc.}$$

Hence $$\frac{u'-u}{h}=4ax^3-6ax^2h+, \text{ etc.}$$

If now we make h equal to zero, Art. 174, all the terms in the second member of this equation except the first disappear, and we have

$$\frac{du}{dx}=4ax^3.$$

or $$du=4ax^3dx;$$

that is, the differential of ax^4 is equal to the differential of x^4 multiplied by a.

PROPOSITION VI.—THEOREM.

(177.) *The differential of a constant term is zero; hence a constant quantity connected with a variable by the sign plus or minus, will disappear in differentiation.*

Suppose we have the function

$$u=b+x^4.$$

When x becomes $x+h$, we have

$$u'=b+x^4+4x^3h+6x^2h^2+, \text{ etc.},$$

and $$u'-u=4x^3h+6x^2h^2+, \text{ etc.}$$

Hence $$\frac{u'-u}{h}=4x^3+6x^2h+, \text{ etc.};$$

and, making h equal to zero, Art. 174, we have

$$\frac{du}{dx}=4x^3,$$

or $$du=4x^3dx,$$

where the constant term b has disappeared in differentiation.

Ex. 1. What is the differential of $5ax^3$?

Ans. $15ax^2dx$.

Ex. 2. What is the differential of $\frac{3}{4}x^3+b$?

Ans.

Ex. 3. What is the differential of $3x^5$?

Ans.

Ex. 4. What is the differential of $7a^2x^6+b^3$?

Ex. 5. What is the differential of $4ab^3x^5-c$?

Ex. 6. What is the differential of $3a^2cx^9-d$?

PROPOSITION VII.—THEOREM.

(178.) *If* u *represents any function of* x, *and we change* x *into* x+h, *the new value of the function will consist of three parts:*

1*st. The primitive function* u.

2*d. The differential coefficient of the function multiplied by the first power of the increment* h.

3*d. A function of* x *and* h *multiplied by the second power of the increment* h.

We have seen in the preceding Propositions that when u is a function of x, and we change x into $x+h$, the new value of the function consists of a series of terms which may be arranged in the order of the ascending powers of h, and the development is of the following form

$$u'=\mathrm{A}+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+, \text{ etc.}$$

Now when we suppose h equal to zero, the second member of this equation reduces to A, and u' on this supposition becomes u; hence

$$\mathrm{A}=u,$$

and the development may be written

$$u'=u+\mathrm{B}h+\mathrm{C}h^2+\mathrm{D}h^3+, \text{ etc.},$$

or $$u'=u+\mathrm{B}h+h^2(\mathrm{C}+\mathrm{D}h+, \text{ etc.}).$$

If now we represent $\mathrm{C}+\mathrm{D}h+$, etc., by C', where C' is a function both of x and h, we have

$$u'=u+\mathrm{B}h+\mathrm{C}'h^2. \qquad (1)$$

Transposing and dividing by h, we find

$$\frac{u'-u}{h}=B+C'h,$$

and when we make h equal to zero, we have

$$\frac{du}{dx}=B;$$

that is, B is the differential coefficient of the function.

We see from equation (1) that u', the new value of the function, consists of the primitive function u, plus the differential coefficient of the function multiplied by h, plus a function of x and h multiplied by h^2.

This new value of the function will be frequently referred to hereafter under the form

$$u'=u+Ah+Bh^2. \qquad (2)$$

PROPOSITION VIII.—THEOREM.

(179.) *The differential of the sum or difference of any number of functions dependent on the same variable, is equal to the sum or difference of their differentials taken separately.*

Let us suppose the function u to be composed of several variable terms, as, for example,

$$u=y+z-v,$$

where y, z, and v are functions of x.

If we change x into $x+h$, we shall have

$$u'-u=(y'-y)+(z'-z)-(v'-v).$$

But by the preceding Proposition $y'-y$ may be put under the form of $Ah+Bh^2$.

So, also, $z'-z$ may be put under the form of $A'h+B'h^2$; and $v'-v$ may be represented by $A''h+B''h^2$; that is,

$$u'-u=(Ah+Bh^2)+(A'h+B'h^2)-(A''h+B''h^2).$$

Dividing each member by h, we have

$$\frac{u'-u}{h}=(A+Bh)+(A'+B'h)-(A''+B''h);$$

and making h equal to zero, Art. 174, we have

$$\frac{du}{dx}=A+A'-A'',$$

or

$$du=Adx+A'dx-A''dx.$$

But Adx is the differential of y; A$'dx$ is the differential of z; and A$''dx$ is the differential of v. Hence

$$du=dy+dz-dv.$$

Ex. 1. What is the differential of $6x^4-5x^3-2x$?

Ans. $(24x^3-15x^2-2)dx$.

Ex. 2. What is the differential of ax^3-cx?

Ans.

Ex. 3. What is the differential of $3ax^3-bx^4$?

Ans.

Ex. 4. What is the differential of $a^6+3a^4x^2+3a^2x^4+x^6$?

Ans.

Ex. 5. What is the differential of $5x^3-3x^2+6x+2$?

Ans.

Ex. 6. What is the differential of $7x^4+6x^3-5ax^2+3x-6$?

Ans.

PROPOSITION IX.—THEOREM.

(180.) *The differential of the product of two functions dependent on the same variable, is equal to the sum of the products obtained by multiplying each by the differential of the other.*

Let us designate two functions by u and v, and suppose them to depend on the same variable x; then, when x is increased so as to become $x+h$, the new functions may be written, Art. 178,

$$u'=u+\mathrm{A}h+\mathrm{B}h^2,$$
$$v'=v+\mathrm{A}'h+\mathrm{B}'h^2.$$

If we multiply together the corresponding members of these equations, we shall have

$$\begin{aligned}u'v'=uv&+\mathrm{A}vh+\mathrm{B}vh^2,\\ &+\mathrm{A}'uh+\mathrm{AA}'h^2+\text{, etc.},\\ &+\mathrm{B}'uh^2+\text{, etc.},\end{aligned}$$

where, it will be observed, the terms omitted contain powers of the increment higher than h^2.

Transposing, and dividing by h, we have

$$\frac{u'v'-uv}{h}=\mathrm{A}v+\mathrm{A}'u+\text{ other terms involving } h.$$

When we make h equal to zero, Art. 174, the terms involving h disappear, and we have

$$\frac{d(uv)}{dx}=\mathrm{A}v+\mathrm{A}'u;$$

or, multiplying by dx,

$$d(uv)=v\mathrm{A}dx+u\mathrm{A}'dx.$$

But $\mathrm{A}dx$ is equal to du,

and $\mathrm{A}'dx$ is equal to dv.

Hence $$d(uv)=vdu+udv, \quad (1)$$

which was the proposition to be demonstrated.

(181.) *Cor.* If we divide both members of equation (1) by uv, we shall have

$$\frac{d(uv)}{uv}=\frac{du}{u}+\frac{dv}{v};$$

that is, *the differential of the product of two functions, divided by their product, is equal to the sum of the quotients obtained by dividing the differential of each function by the function itself.*

Ex. 1. What is the differential of xy^2?

Ans. $y^2dx+2xydy$.

Ex. 2. What is the differential of x^2y^2?

Ans.

Ex. 3. What is the differential of ax^2y^2?

Ans.

Ex. 4. What is the differential of $ax^2(x^2+2b)$?

Ans.

Ex. 5. What is the differential of $(x^2+a)(2x+b)$?

Ans.

Ex. 6. What is the differential of $(x^2+a)(3x^2+b)$?

Ans.

PROPOSITION X.—THEOREM.

(182.) *The differential of the product of any number of functions of the same variable, is equal to the sum of the products obtained by multiplying the differential of each function by the product of the others.*

Let us designate three functions by u, v, and z, and suppose them to depend on the same variable x. Substitute y for vz, and we shall have

$$uvz=uy,$$

and $$d(uvz)=d(uy).$$

But, by the preceding Proposition,

$$d(uy)=ydu+udy; \qquad (1)$$

and since $y=vz$, we have, by the same Proposition,

$$dy=zdv+vdz.$$

Substituting these values of y and dy in equation (1), it becomes

$$d(uvz)=vzdu+uzdv+uvdz. \qquad (2)$$

The same method is applicable to the product of four or more functions.

(183.) *Cor.* If we divide both members of equation (2) by uvz, we shall have

$$\frac{d(uvz)}{uvz}=\frac{du}{u}+\frac{dv}{v}+\frac{dz}{z},$$

which is an extension of Art. 181.

Ex. 1. What is the differential of xy^2z?

Ans. $y^2zdx+2xyzdy+xy^2dz$.

Ex. 2. What is the differential of $x^2y^3z^2$?

Ans.

Ex. 3. What is the differential of axy^2z^3?

Ans.

Ex. 4. What is the differential of $x(x^2+a)(x+2b)$?

Ans.

Ex. 5. What is the differential of $ax^2(x^3+a)(x+3b)$?

Ans.

PROPOSITION XI.—THEOREM.

(184.) *The differential of a fraction is equal to the denominator into the differential of the numerator, minus the numerator into the differential of the denominator, divided by the square of the denominator.*

Let us designate the fraction by $\frac{u}{v}$, and suppose

$$\frac{u}{v}=y, \qquad (1)$$

then $$u=vy.$$

Therefore, by Prop. IX.,

$$du = ydv + vdy;$$

whence $$vdy = du - ydv. \tag{2}$$

Substituting in the second member of equation (2) the value of y from equation (1), we have

$$vdy = du - \frac{udv}{v}.$$

Dividing by v, we obtain

$$dy = \frac{vdu - udv}{v^2};$$

that is, $$d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2},$$

which was the proposition to be demonstrated.

(185.) *Cor.* If the numerator u is constant, its differential will be zero, Art. 177, and we shall have

$$d\left(\frac{c}{v}\right) = \frac{-cdv}{v^2}.$$

Ex. 1. What is the differential of $\frac{x^2}{y^3}$?

Ans. $\frac{2xy^3dx - 3x^2y^2dy}{y^6}$, or $\frac{2xydx - 3x^2dy}{y^4}$.

Ex. 2. What is the differential of $\frac{a}{x^2}$?

Ans.

Ex. 3. What is the differential of $\frac{c}{ax^3}$?

Ans.

Ex. 4. What is the differential of $\frac{x}{1-x}$?

Ans.

Ex. 5. What is the differential of $\frac{1+x^2}{1-x^2}$?

Ans.

Ex. 6. What is the differential of $\frac{a^2+x^2}{b^2-x^2}$?

Ans.

PROPOSITION XII.—THEOREM.

(186.) *To obtain the differential of a variable affected with any exponent whatever, we must diminish the exponent of the power by unity, and then multiply by the primitive exponent and by the differential of the variable.*

This is the same as Prop. IV., and the demonstration there given, being founded on the binomial theorem, may be considered sufficiently general, since the binomial theorem is true, whether the exponent of the power be positive or negative, integral or fractional. This theorem may, however, be deduced directly from Prop. X.

Let it be required to find the differential of x^n, where the exponent n may be either positive or negative, integral or fractional.

Case first. When n is a positive and whole number.

x^n may be considered as the product of n factors each equal to x. Hence, by Prop. X., Cor.,

$$\frac{d(x^n)}{x^n}=\frac{d(xxxx\ldots.)}{xxxx\ldots.}=\frac{dx}{x}+\frac{dx}{x}+\frac{dx}{x}+\frac{dx}{x}+\ldots.;$$

and since there are n equal factors in the first member of the equation, there will be n equal terms in the second; hence

$$\frac{d(x^n)}{x^n}=\frac{ndx}{x},$$

or

$$d(x^n)=nx^{n-1}dx.$$

Case second. When n is a positive fraction.

Represent the fraction by $\frac{r}{s}$, and let

$$u=x^{\frac{r}{s}}.$$

Raising both members to the power s, we shall have

$$u^s=x^r,$$

and, since r and s are supposed to represent entire numbers, we shall have, by the first Case,

$$su^{s-1}du=rx^{r-1}dx;$$

whence we find

$$du=\frac{rx^{r-1}}{su^{s-1}}dx=\frac{rx^{r-1}}{sx^{\frac{r}{s}(s-1)}}dx,$$

which may be reduced to

$$du=\frac{r}{s}x^{\frac{r}{s}-1}dx,$$

which is of the same form as $nx^{n-1}dx$, substituting $\frac{r}{s}$ for n.

Case third. When n is negative, either integral or fractional.

Suppose $$u=x^{-n},$$

which may be written $$u=\frac{1}{x^n}.$$

Differentiating by Prop. XI., Cor., we have

$$du=\frac{-d(x^n)}{x^{2n}};$$

and differentiating the numerator by Case first, or by Case second, if n represents a fraction, we have

$$du=\frac{-nx^{n-1}dx}{x^{2n}};$$

or, subtracting the exponent $2n$ from $n-1$, we have

$$du=-nx^{-n-1}dx,$$

which is of the same form as $nx^{n-1}dx$, by substituting $-n$ for $+n$.

Proposition XII. may, therefore, be considered general, whatever be the exponent of x.

Ex. 1. What is the differential of ax^{n+1}?

Ans. $a(n+1)x^n dx$.

Ex. 2. What is the differential of $\frac{a}{b}x^{\frac{1}{n}}+c$?

Ans.

Ex. 3. What is the differential of $ab^2x^{\frac{2}{3}}$?

Ans.

Ex. 4. What is the differential of $bx^{\frac{2}{5}}$?

Ans.

Ex. 5. What is the differential of cx^{-3}?

Ans.

Ex. 6. What is the differential of $x^2y^{\frac{1}{2}}z^3$?

Ans.

Ex. 7. If the area of a square increase uniformly at the rate of $\frac{1}{10}$ of a square inch per second, at what rate is the side increasing when the area is 100 square inches?

Ans.

Ex. 8. If the solidity of a cube increase uniformly at the rate of a cubic inch per second, at what rate is the edge increasing when the solid becomes a cubic foot?

Ans.

PROPOSITION XIII.—THEOREM.

(187.) *The differential of the square root of a variable quantity, is equal to the differential of that quantity divided by twice the radical.*

Let it be required to find the differential of

$$\sqrt{x}, \text{ or } x^{\frac{1}{2}}.$$

According to the preceding Proposition,

$$d(x^{\frac{1}{2}})=\tfrac{1}{2}x^{\frac{1}{2}-1}dx,$$
$$=\tfrac{1}{2}x^{-\frac{1}{2}}dx,$$

which may be written $\dfrac{dx}{2\sqrt{x}}$.

Ex. 1. What is the differential of $\sqrt{ax^3}$?

Ans. $\dfrac{3ax^2dx}{2\sqrt{ax^3}}$, or $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}}dx$.

Ex. 2. What is the differential of $\sqrt{abx^7}$?

Ans.

Ex. 3. What is the differential of $\sqrt{ax^6}$?

Ans.

Ex. 4. What is the differential of $a\sqrt{x}-\dfrac{x}{3}$?

Ans.

Ex. 5. What is the differential of $\sqrt{ax}+\sqrt{c^2x^3}$?

Ans.

PROPOSITION XIV.—THEOREM.

(188.) *To obtain the differential of a polynomial raised to any power, we must diminish the exponent of the power by unity, and then multiply by the primitive exponent and by the differential of the polynomial.*

Let it be required to differentiate the function

$$u=(ax+x^2)^n.$$

Substitute y for $ax+x^2$, and we have

$$u=y^n.$$

Whence, by Prop. XII., $du=ny^{n-1}dy$.

Restoring the value of y, we have

$$du=n(ax+x^2)^{n-1}d(ax+x^2),$$

which is conformable to the Proposition.

The differentiation of $ax+x^2$ is here only indicated. If we actually perform it, we shall have

$$du=n(ax+x^2)^{n-1}(a+2x)dx.$$

Ex. 1. What is the differential of $\sqrt{a+bx^2}$?

Ans. $\dfrac{bxdx}{\sqrt{a+bx^2}}$.

Ex. 2. What is the differential of $(ax^2+x^3)^3$?

Ans.

Ex. 3. What is the differential of $\sqrt{ax+bx^2+cx^3}$?

Ans.

Ex. 4. What is the differential of $(ax-x^3)^5$?

Ans.

Ex. 5. What is the differential of $(a+bx^2)^{\frac{5}{4}}$?

Ans.

Ex. 6. What is the differential of $(a+x^2)^{\frac{1}{n}}$?

Ans.

Ex. 7. If x increase uniformly at the rate of $\frac{1}{100}$ of an inch per second, at what rate is the expression $(1+x)^3$ increasing when x equals 9 inches? *Ans.*

(189.) By the application of the preceding principles, complicated algebraic functions may be differentiated.

Ex. 8. What is the differential of the function $u=\dfrac{a+x}{a+x^2}$?

According to Prop. XI.,

$$du=\frac{(a+x^2)dx-2x(a+x)dx}{(a+x^2)^2},$$

which may be reduced to

$$du=\frac{(a-2ax-x^2)dx}{(a+x^2)^2}.$$

Ex. 9. Differentiate the function $u=\sqrt{x^2+a\sqrt{x}}$.

Ans.

Ex. 10. Differentiate the function $u=\frac{(b+x)^3}{x}$.

Ans.

Ex. 11. Differentiate the function $u=\frac{x^3}{(a+x^2)^2}$.

Ans.

Ex. 12. Differentiate the function $u=\frac{1}{(a+x^2)^n}$.

Ans.

Ex. 13. If the side of an equilateral triangle increase uniformly at the rate of half an inch per second, at what rate is its perpendicular increasing when its side is equal to 8 inches?

Ans. $\frac{\sqrt{3}}{4}$ inch per second.

Ex. 14. If the side of an equilateral triangle increase uniformly at the rate of half an inch per second, at what rate is the area increasing when the side becomes 8 inches?

Ans. $2\sqrt{3}$ inches per second.

Ex. 15. If a circular plate of metal expand by heat so that its diameter increases uniformly at the rate of $\frac{1}{100}$ of an inch per second, at what rate is its surface increasing when the diameter is exactly two inches?

Ans. $\frac{\pi}{100}$ inch per second.

Ex. 16. If a circular plate expand so that its area increases uniformly at the rate of $\frac{1}{50}$ of a square inch per second, at what rate is its diameter increasing when the area of the circle is exactly a square inch?

Ans. $\frac{1}{50\sqrt{\pi}}$ inch per second.

Ex. 17. If the diameter of a spherical soap bubble increases uniformly at the rate of $\frac{1}{10}$ of an inch per second, at what rate is its capacity increasing at the moment the diameter becomes two inches?

Ans. $\frac{\pi}{5}$ inch per second.

Ex. 18. If the capacity of a spherical soap bubble increases uniformly at the rate of two cubic inches per second, at what

rate is the diameter increasing at the moment it becomes two inches?

Ans. $\frac{1}{\pi}$ inch per second.

Ex. 19. A boy standing on the top of a tower whose height is 60 feet, observed another boy running toward the foot of the tower at the rate of five miles an hour on the horizontal plane; at what rate is he approaching the first when he is 80 feet from the foot of the tower?

Ans. 4 miles an hour.

Ex. 20. If the diameter of a circular plate expand uniformly at the rate of $\frac{1}{10}$ of an inch per second, what is the diameter of the circle when its area is expanding at the rate of a square inch per second?

Ans. $\frac{20}{\pi}$ inches.

Ex. 21. If the diameter of a sphere increase uniformly at the rate of $\frac{1}{10}$ of an inch per second, what is its diameter when the capacity is increasing at the rate of five cubic inches per second?

Ans. $\frac{10}{\sqrt{\pi}}$ inches.

Ex. 22. If the diameter of the base of a cone increase uniformly at the rate of $\frac{1}{10}$ inch per second, at what rate is its solidity increasing when the diameter of the base becomes 10 inches, the height being constantly one foot?

Ans. 2π inches per second.

SECTION II.

OF SUCCESSIVE DIFFERENTIALS—MACLAURIN'S THEOREM—TAYLOR'S THEOREM—FUNCTIONS OF SEVERAL INDEPENDENT VARIABLES.

(190.) Since the differentials of all expressions which contain x raised to any power, also contain x raised to the next inferior power, Art. 186, we may consider the differential coefficient of a function as a new function, and determine its differential accordingly. We thus obtain the *second differential coefficient.*

For example, if

$$u=ax^3,$$

$$\frac{du}{dx}=3ax^2.$$

Now since $3ax^2$ contains x, we may differentiate it as a new function, and we obtain

$$d\left(\frac{du}{dx}\right)=6axdx.$$

But, since dx is supposed to be a constant,

$$d\left(\frac{du}{dx}\right)=\frac{d(du)}{dx}=\frac{d^2u}{dx};$$

the symbol d^2u (which is read *second differential of* u) being used to indicate that the function u has been differentiated twice, or that we have taken the differential of the differential of u. Hence

$$\frac{d^2u}{dx}=6axdx;$$

or, dividing each side by dx,

$$\frac{d^2u}{dx^2}=6ax,$$

where dx^2 represents *the square of the differential of* x, and not the differential of x^2.

The expression $6ax$ being the differential coefficient of the

first differential coefficient, is called the *second differential coefficient.*

Again, since $6ax$ contains x, we may differentiate it as a new function, and we obtain

$$\frac{d^3u}{dx^2}=6adx;$$

or, dividing each side by dx,

$$\frac{d^3u}{dx^3}=6a,$$

which is the differential coefficient of the second differential coefficient, and is called the *third differential coefficient.*

The third differential coefficient $\frac{d^3u}{dx^3}$ is read third differential of u, divided by the cube of the differential of x.

As the expression $6a$ does not contain x, the differentiation can be carried no further, and we find the function $u=ax^3$ has three differential coefficients. Other functions may have a greater number of differential coefficients.

The learner must not confound d^2u with du^2, the former denoting the *differential of the differential of* u, and the latter the *square of the differential of* u.

Ex. 1. Determine the successive differentials of ax^4.

Ex. 2. Determine the successive differentials of $(a+x^2)^2$.

MACLAURIN'S THEOREM.

(191.) Maclaurin's theorem explains the method of developing into a series any function of a single variable.

PROPOSITION I.—THEOREM OF MACLAURIN.

If u *represent a function of* x *which it is possible to develop in a series of positive ascending powers of that variable, then will that development be*

$$u=(u)+\left(\frac{du}{dx}\right)x+\left(\frac{d^2u}{dx^2}\right)\frac{x^2}{2}+\left(\frac{d^3u}{dx^3}\right)\frac{x^3}{2.3}+, \text{ etc.},$$

where the brackets indicate the values which the inclosed functions assume when x equals zero.

Let u represent any function of x, as, for example, $(a+x)^n$, and let us suppose that this function, when expanded, will con-

tain the ascending powers of x, and coefficients not containing x, which are to be determined. Let these coefficients be represented by A, B, C, etc., then we shall have

$$u=A+Bx+Cx^2+Dx^3+Ex^4+, \text{ etc.} \qquad (1)$$

If we differentiate this equation, and divide both sides by dx, we obtain

$$\frac{du}{dx}=B+2Cx+3Dx^2+4Ex^3+, \text{ etc.}$$

If we continue to differentiate, and divide by dx, it is obvious that the coefficients A, B, C, etc., will disappear in succession, and the result will be as follows:

$$\frac{d^2u}{dx^2}=2C+2.3Dx+\ 3.4Ex^2+, \text{ etc.},$$

$$\frac{d^3u}{dx^3}=\quad 2.3D\ +2.3.4Ex\ +, \text{ etc.},$$

etc., etc.

Represent by (u) what u becomes when $x=0$.

Represent by $\left(\frac{du}{dx}\right)$ what $\frac{du}{dx}$ becomes when $x=0$.

Represent by $\left(\frac{d^2u}{dx^2}\right)$ what $\frac{d^2u}{dx^2}$ becomes when $x=0$,

and so on; the preceding equations furnish us

$$(u)\ =A,$$

$$\left(\frac{du}{dx}\right)=B,$$

$$\left(\frac{d^2u}{dx^2}\right)=2C,$$

$$\left(\frac{d^3u}{dx^3}\right)=2.3D;$$

whence we see

$$A=(u),\ B=\left(\frac{du}{dx}\right),\ C=\frac{1}{2}\left(\frac{d^2u}{dx^2}\right),\ D=\frac{1}{2.3}\left(\frac{d^3u}{dx^3}\right), \text{ etc.}$$

Substituting these values in equation (1), it becomes

$$u=(u)+\left(\frac{du}{dx}\right)x+\left(\frac{d^2u}{dx^2}\right)\frac{x^2}{2}+\left(\frac{d^3u}{dx^3}\right)\frac{x^3}{2.3}, \text{ etc.},$$

which is Maclaurin's theorem.

(192.) Ex. 1. Expand $(a+x)^n$ into a series.

When $x=0$, this function reduces to a^n.

Hence $\qquad (u)=a^n$.

By differentiation, we obtain

$$\frac{du}{dx}=n(a+x)^{n-1},$$

which becomes, when $x=0$,

$$na^{n-1}.$$

Hence
$$\left(\frac{du}{dx}\right)=na^{n-1}.$$

Also,
$$\frac{d^2u}{dx^2}=n(n-1)(a+x)^{n-2},$$

which becomes, when $x=0$,

$$n(n-1)a.$$

Also,
$$\frac{d^3u}{dx^3}=n(n-1)(n-2)(a+x)^{n-3},$$

which becomes, when $x=0$,

$$n(n-1)(n-2)a^{n-3}.$$

Substituting these values in Maclaurin's theorem, we have

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+, \text{ etc.},$$

which is the same as found by the Binomial theorem.

Ex. 2. Develop into a series the function $u=\frac{1}{a+x}$.

By differentiation, we find

$$\frac{du}{dx}=-\frac{1}{(a+x)^2},$$
$$\frac{d^2u}{dx^2}=\frac{2}{(a+x)^3},$$
$$\frac{d^3u}{dx^3}=-\frac{2.3}{(a+x)^4}.$$

Making $x=0$, in the values of u, of $\frac{du}{dx}$, of $\frac{d^2u}{dx^2}$, of $\frac{d^3u}{dx^3}$, etc., we find

$$(u)=\frac{1}{a},\ \left(\frac{du}{dx}\right)=-\frac{1}{a^2},\ \left(\frac{d^2u}{dx^2}\right)=\frac{2}{a^3},\ \left(\frac{d^3u}{dx^3}\right)=-\frac{2.3}{a^4}, \text{ etc.}$$

Substituting these values in Maclaurin's theorem, we obtain

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{x^2}{a^3}-\frac{x^3}{a^4}+, \text{ etc.}$$

Ex. 3. Develop into a series the function $\frac{1}{1-x}$.

Ans. $1+x+x^2+x^3+x^4+$, etc.

Ex. 4. Develop into a series the function $\sqrt{a+x}$.

Ans. $a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{2.4}a^{-\frac{3}{2}}x^2+\frac{1.3}{2.4.6}a^{-\frac{5}{2}}x^3-$, etc.

Ex. 5. Expand into a series $(a-x)^{-2}$.

Ans.

Ex. 6. Expand into a series $(a+x)^{-3}$.

Ans. $a^{-3}-3a^{-4}x+6a^{-5}x^2-10a^{-6}x^3+15a^{-7}x^4-$, etc.

(193.) When in the application of Maclaurin's theorem the variable x is made equal to 0, the function u, or some of its differential coefficients, may become infinite. Such functions can not be developed by Maclaurin's theorem.

Thus, if we have

$$u=\log. x, \quad u=\cot. x, \text{ or } u=\frac{1}{x},$$

when we make $x=0$, u becomes equal to infinity.

Also, if we have $\qquad u=ax^{\frac{1}{2}},$

the first differential coefficient is

$$\frac{du}{dx}=\frac{a}{2x^{\frac{1}{2}}},$$

which becomes infinite when x is made equal to zero.

Hence neither of these functions can be developed by Maclaurin's theorem.

TAYLOR'S THEOREM.

(194.) Taylor's theorem explains the method of developing into a series a function of the sum or difference of two variables.

The following principle is assumed in the demonstration of Taylor's theorem.

PROPOSITION II.—THEOREM.

If we have a function of the sum or difference of two variables, x *and* y, *the differential coefficient will be the same, if we suppose* x *to vary and* y *to remain constant, as when we suppose* y *to vary and* x *to remain constant.*

Thus, let $\qquad u=(x+y)^n.$

If we suppose x to vary and y to remain constant, we have

$$\frac{du}{dx}=n(x+y)^{n-1};$$

and if we suppose y to vary and x to remain constant, we have

$$\frac{du}{dy}=n(x+y)^{n-1},$$

the same as under the first supposition.

PROPOSITION III.—TAYLOR'S THEOREM.

(195.) *Any function of the sum of two variables may be developed into a series of the following form:*

$$F(x+y)=u+\frac{du}{dx}y+\frac{d^2u}{dx^2}\frac{y^2}{1.2}+\frac{d^3u}{dx^3}\frac{y^3}{1.2.3}+,\text{ etc.,}$$

where u represents the value of the function when $y=0$.

Let u' be a function of $x+y$, which we will suppose to be developed into a series, and arranged according to the powers of y, so that we have

$$u'=F(x+y)=\text{A}+\text{B}y+\text{C}y^2+\text{D}y^3+,\text{ etc.,} \quad (1)$$

where A, B, C, D, etc., are independent of y, but dependent upon x and upon the constants which enter the primitive function. It is now required to find such values for A, B, C, D, etc., as shall render the development true for all possible values which may be attributed to x and y.

If we differentiate under the supposition that x varies and y remains constant, we shall have

$$\frac{du'}{dx}=\frac{d\text{A}}{dx}+\frac{d\text{B}}{dx}y+\frac{d\text{C}}{dx}y^2+\frac{d\text{D}}{dx}y^3+,\text{ etc.}$$

If we differentiate under the supposition that y varies and x remains constant, we shall have

$$\frac{du'}{dy}=\text{B}+2\text{C}y+3\text{D}y^2+,\text{ etc.}$$

But, by the preceding Proposition,

$$\frac{du'}{dx}=\frac{du'}{dy}.$$

Hence we must have

$$\frac{d\text{A}}{dx}+\frac{d\text{B}}{dx}y+\frac{d\text{C}}{dx}y^2+,\text{ etc.,}=\text{B}+2\text{C}y+3\text{D}y^2+,\text{ etc.,}$$

and since the coefficients of the series are independent of y, and the equality exists whatever be the value of y, it follows that the terms involving the same powers of y in the two members are respectively equal (Algebra, Art. 300). Therefore,

$$\frac{dA}{dx}=B, \qquad (2)$$

$$\frac{dB}{dx}=2C, \qquad (3)$$

$$\frac{dC}{dx}=3D, \text{ etc.} \qquad (4)$$

If in equation (1) we make $y=0$, function of $x+y$ will reduce to function of x, which we will denote by u. Therefore

$$A=u.$$

Substituting this value of A in equation (2), we have

$$B=\frac{du}{dx}.$$

Substituting this value of B in equation (3), we have

$$2C=\frac{d^2u}{dx^2}; \text{ whence } C=\frac{d^2u}{2dx^2}.$$

Substituting this value of C in equation (4), we have

$$3D=\frac{d^3u}{2dx^3}; \text{ whence } D=\frac{d^3u}{2.3dx^3}.$$

Substituting these values of A, B, C, D in equation (1), we have Taylor's formula,

$$u'=F(x+y)=u+\frac{du}{dx}y+\frac{d^2u}{dx^2}\frac{y^2}{1.2}+\frac{d^3u}{dx^3}\frac{y^3}{1.2.3}+, \text{ etc.},$$

where the first term of the series, u, represents what the function to be developed becomes when the variable, according to the ascending powers of which the series is arranged, is made equal to zero.

In a similar manner, we find the development of $F(x-y)$ to be

$$u'=F(x-y)=u-\frac{du}{dx}y+\frac{d^2u}{dx^2}\frac{y^2}{1.2}-\frac{d^3u}{dx^3}\frac{y}{1.2.3}+ \text{ etc.}$$

(196.) Ex. 1. Required the development of the function

$$u'=(x+y)^n.$$

Making $y=0$, we obtain $u=x^n$, and thence, by differentiation,

$$\frac{du}{dx}=nx^{n-1},\quad \frac{d^2u}{dx^2}=n(n-1)x^{n-2},$$

$$\frac{d^3u}{dx^3}=n(n-1)(n-2)x^{n-3},\ \text{etc.}$$

These values being substituted in the formula, give

$$u'=(x+y)^n=x^n+nx^{n-1}y+\frac{n(n-1)}{1.2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}y^3+,$$
etc.

the same as found by the Binomial theorem.

Ex. 2. Required the development of the function

$$u'=\sqrt{x+y}.$$

Ans. $u'=(x+y)^{\frac{1}{2}}=x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}y-\frac{1}{2.4}x^{-\frac{3}{2}}y^2+\frac{1.3}{2.4.6}x^{-\frac{5}{2}}y^3-$, etc.

Ex. 3. Required the development of the function

$$u'=\sqrt[3]{x+y}.$$

Ans. $u'=(x+y)^{\frac{1}{3}}=x^{\frac{1}{3}}+\frac{1}{3}x^{-\frac{2}{3}}y-\frac{2}{3.6}x^{-\frac{5}{3}}y^2+\frac{2.5}{3.6.9}x^{-\frac{8}{3}}y^3-$, etc.

(197.) Although the general development of every function of $x+y$ is correctly given by Taylor's theorem, particular values may sometimes be assigned to the variables which shall render this form of development impossible; and this impossibility will be indicated by some of the coefficients in Taylor's theorem becoming infinite. Thus, if we have

$$u'=a+(b-x+y)^{\frac{1}{2}},$$

$$u=a+(b-x)^{\frac{1}{2}}.$$

Therefore,
$$\frac{du}{dx}=-\frac{1}{2(b-x)^{\frac{1}{2}}},$$

and
$$\frac{d^2u}{dx^2}=-\frac{1}{4(b-x)^{\frac{3}{2}}}.$$

etc., etc.,

in which all the differential coefficients will become equal to infinity, when we make $x=b$.

So, also, if we have

$$u'=a+(b-x+y)^{\frac{1}{n}},$$

in which n is a whole number, all the differential coefficients will become infinite when we make $x=b$.

The supposition that $x=b$ reduces the above equation to

$$u' = a + y^{\frac{1}{2}},$$

and $$u = a,$$

where u' and u are expressed under different forms; and, in general, when the proposed function changes its form by attributing particular values to the variables, the development can not be made by Taylor's theorem.

DIFFERENTIATION OF FUNCTIONS OF TWO OR MORE INDEPENDENT VARIABLES.

(198.) Let u be a function of two independent variables x and y; then, since in consequence of this independence, however either be supposed to vary, the other will remain unchanged, the function ought to furnish two differential coefficients; the one arising from ascribing a variation to x, and the other from ascribing a variation to y; y entering the first coefficient as if it were a constant, and x entering the second as if it were a constant.

If we suppose y to remain constant and x to vary, the differential coefficient will be

$$\frac{du}{dx};$$

and if we suppose x to remain constant and y to vary, the differential coefficient will be

$$\frac{du}{dy}.$$

(199.) The differential coefficients which are obtained under these suppositions are called *partial differential coefficients.* The first is the partial differential coefficient with respect to x, and the second with respect to y.

If we multiply the several partial differential coefficients by dx and dy, we obtain

$$\frac{du}{dx}dx, \ \frac{du}{dy}dy;$$

which are called *partial differentials;* the first is a partial differential with respect to x, and the second a partial differential with respect to y.

The differential which is obtained under the supposition that both the variables have changed their values, is called the *total differential* of the function; that is,

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy.$$

(200.) If we have a function of three variables, x, y, and z, we should necessarily have as many independent differentials, of which the aggregate would be the total differential of the function; that is,

$$du=\frac{du}{dx}dx+\frac{du}{dy}dy+\frac{du}{dz}dz.$$

Hence, whether the variables are dependent or independent, we conclude that *the total differential of a function of any number of variables is the sum of the several partial differentials, arising from differentiating the function relatively to each variable in succession, as if all the others were constants.*

Ex. 1. If one side of a rectangle increase at the rate of 1 inch per second, and the other at the rate of 2 inches, at what rate is the area increasing when the first side becomes 8 inches, and the last 12 inches?

Ans. 28 inches per second.

Ex. 2. If one side of a rectangle increase at the rate of 2 inches per second, and the other diminish at the rate of 3 inches per second, at what rate is the area increasing or diminishing when the first side becomes 10 inches, and the second 8?

Ans.

Ex. 3. If the major axis of an ellipse increase uniformly at the rate of 2 inches per second, and the minor axis at the rate of 3 inches, at what rate is the area increasing when the major axis becomes 20 inches, the minor axis at the same instant being 12 inches?

Ans. 21π inches.

Ex. 4. If the altitude of a cone diminishes at the rate of 3 inches per second, and the diameter of the base increases at the rate of 1 inch per second, at what rate does the solidity vary when the altitude becomes 18 inches, the diameter of the base at the same instant being 10 inches?

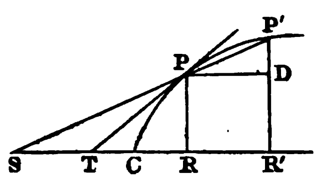
Ans.

SECTION III.

SIGNIFICATION OF THE FIRST DIFFERENTIAL COEFFICIENT—MAXIMA AND MINIMA OF FUNCTIONS.

PROPOSITION I.—THEOREM.

(201.) *The tangent of the angle which a tangent line at any point of a curve makes with the axis of abscissas, is equal to the first differential coefficient of the ordinate of the curve.*



Let CPP' be a curve, and P any point of it whose co-ordinates are x and y. Increase the abscissa CR or x by the arbitrary increment RR', which we will represent by h; denote the corresponding ordinate P'R' by y', and draw the secant line SPP'. Then

$$P'D = P'R' - PR = y' - y.$$

But from the triangle PDP' we have

$$PD : P'D :: 1 : \text{tang. } S = \frac{P'D}{PD};$$

and, substituting for P'D and PD their values, we have

$$\frac{y'-y}{h} = \text{tang. } S,$$

which expresses the ratio of the increment of y to that of x. In order to find the differential coefficient of y with respect to x, we must find the limit of this ratio by making the increment equal to zero, Art. 174.

Now if h be diminished, the point P' approaches P, the secant SP approaches the tangent TP; and, finally, when $h=0$, the point P' coincides with P, and the secant with the tangent. In this case we have

$$\frac{dy}{dx} = \text{tang. } T.$$

(202.) If it is required to find the point of a given curve at

which the tangent line makes a given angle with the axis of X, we must put the first differential coefficient equal to the tangent of the given angle. If we represent this tangent by a, we must have

$$\frac{dy}{dx}=a;$$

and this, combined with the equation of the curve, will give the values of x and y for the required point.

Ex. It is required to find the point on a parabola, at which the tangent line makes an angle of 45° with the axis.

The equation of the parabola, Art. 50, is

$$y^2=2px.$$

Differentiating, we obtain

$$2ydy=2pdx,$$

or

$$\frac{dy}{dx}=\frac{p}{y}.$$

But since tang. 45° equals radius or unity, we have

$$\frac{p}{y}=1, \text{ or } p=y.$$

Whence, from the equation $y^2=2px$, we find

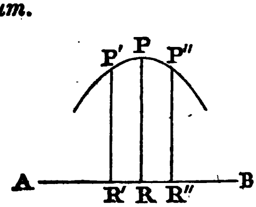
$$x=\frac{p}{2}.$$

Hence the required tangent passes through the extremity of the ordinate drawn from the focus.

OF THE MAXIMA AND MINIMA OF FUNCTIONS OF A SINGLE VARIABLE.

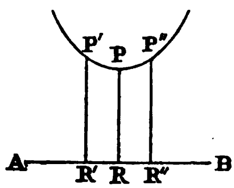
(203.) If a variable quantity gradually increase, and, after it has reached a certain magnitude, gradually decrease, at the end of its increase it is called a *maximum*.

Thus, if a line P′R′, moving from A along AB so as to be always at right angles to AB, gradually increases until it comes into the position PR, and after that gradually decreases, the line is said to be a maximum, or at its greatest value, when it comes into the position PR.



(204.) If a variable quantity gradually decrease, and, after

it has attained a certain magnitude, gradually increases, at the end of its decrease it is called a *minimum.*



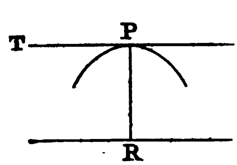
Thus, if a line P′R′, moving from A along AB, gradually decreases until it comes into the position PR, and after that gradually increases, the line is said to be a minimum, or at its least value, when it comes into the position PR.

(205.) If u be a function of x, and if x be decreased by a small quantity so as to give the next preceding value to u, denoted by u', and then increased by the same quantity so as to give the next succeeding value u''; if u be greater than both u' and u'', it will be a maximum; if u be less than both u' and u'', it will be a minimum.

Hence *the maximum value of a variable function exceeds those values which immediately precede and follow it, and the minimum value of a variable function is less than those values which immediately precede and follow it.*

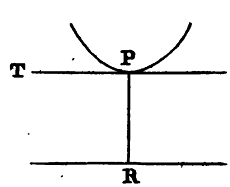
(206.) We have seen, Art. 201, that if y represents the ordinate, and x the abscissa of any curve, the tangent of the angle which the tangent line forms with the axis of abscissas, will be represented by

$$\frac{dy}{dx}.$$



If PR becomes a maximum, the tangent TP, being then parallel with the axis of abscissas, makes no angle with this axis, and we have

$$\frac{dy}{dx}=0.$$



If PR becomes a minimum, the tangent TP, being then parallel with the axis of abscissas, makes no angle with this axis, and we have

$$\frac{dy}{dx}=0.$$

Thus, the equation $\frac{dy}{dx}=0$ simply expresses the condition that the tangent at P is parallel with the axis of abscissas; and,

consequently, the ordinate to that point of the curve must be either at its maximum or minimum value.

(207.) In order, therefore, to determine whether a function has a maximum or a minimum value, we make its first differential coefficient, $\frac{du}{dx}$, equal to zero, and find the value of x in this equation. Represent this root by a. Then substitute successively for x in the given function $a+h$ and $a-h$. If both the results are less than the one obtained by substituting a, this value will be a maximum; if both results are greater, this value will be a minimum.

Ex. 1. Find the value of x which will render u a maximum in the equation

$$u=10x-x^2.$$

Differentiating, we obtain

$$\frac{du}{dx}=10-2x.$$

Putting this differential coefficient equal to zero, we have

$$10-2x=0.$$

Whence $$x=5.$$

Let us now substitute for x in the given function 5, 5−1 and 5+1 successively.

Substituting 4 for x, we have $u'=40-16=24$.
" 5 " x, " $u=50-25=25$.
" 6 " x, " $u''=60-36=24$.

The results of the substitution of 5−1 and 5+1 for x are both less than that obtained by substituting 5. Hence the function u is a maximum when $x=5$.

Ex. 2. Find the value of x which will render u a minimum in the equation

$$u=x^2-16x+70.$$

Differentiating, we obtain

$$\frac{du}{dx}=2x-16.$$

Putting this equal to zero, we have

$$2x-16=0.$$

Whence $$x=8.$$

Let us now substitute for x in the given function 8, 8−1 and 8+1 successively.

Substituting 7 for x, $u' = 49-112+70=7$.
" 8 " x, $u = 64-128+70=6$.
" 9 " x, $u''=81-144+70=7$.

The results of the substitution of 8−1 and 8+1 for x are both greater than that obtained by substituting 6. Hence the function u is a minimum when $x=8$.

(208.) A general method of determining maxima and minima of functions of a single variable, may be deduced from Taylor's theorem, Art. 195.

Suppose we have $u=F(x)$,
and let the variable x be first increased by h, and then diminished by h; and let

$$u'=F(x-h),\ u''=F(x+h);$$

then, by Taylor's theorem, we shall have

$$u''-u= \frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}+\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{, etc.},$$

$$u'-u=-\frac{du}{dx}h+\frac{d^2u}{dx^2}\frac{h^2}{1.2}-\frac{d^3u}{dx^3}\frac{h^3}{1.2.3}+\text{, etc.}$$

Now in order that u may be a maximum, it must be greater than both u' and u''; that is, the second members of the above equations for an infinitely small value of h must be negative; and in order that u may be a minimum, it must be less than both u' and u''; that is, the second member of the above equations for an infinitely small value of h must be positive. Now when h is infinitely small, the signs of the series will be the same as the signs of their first terms. But these terms have contrary signs; hence the function u can have neither a maximum nor a minimum, unless the first term of each series be zero, which requires that

$$\frac{du}{dx}=0,$$

and the roots of this equation will give all the values of x which can render the function u either a maximum or a minimum.

Having made the first differential coefficient equal to zero, the signs of the series will depend on the sign of the second differential coefficient.

If the second differential coefficient is negative, the function is a maximum; if positive, a minimum.

If the second differential coefficient reduces to zero, the signs of the series will again be opposite, and there can neither be a maximum nor a minimum unless the third differential coefficient reduces to zero, in which case the signs will be the same as that of the fourth differential coefficient.

(209.) Hence, in order to find the values of x which will render the proposed function a maximum or a minimum, we have the following

RULE.

Find the first differential coefficient of the function, and place it equal to zero.

Substitute each of the roots of the equation thus formed, in the second differential coefficient. Each one which gives a negative result will, when substituted in the function, make it a maximum, and each which gives a positive result will make it a minimum.

If either root reduces the second differential coefficient to zero, substitute in the third, fourth, etc., until one is found which does not reduce to zero. If this differential coefficient be of an odd order, this root will not render the function either a maximum or a minimum. But if it be of an even order and negative, the function will be a maximum; if positive, a minimum.

Ex. 1. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-3x^2-24x+85.$$

Differentiating, we obtain

$$\frac{du}{dx}=3x^2-6x-24.$$

Placing this equation to zero, we have

$$3x^2-6x-24=0,$$

or

$$x^2-2x-\ 8=0,$$

the roots of which are $+4$ and -2.

The second differential coefficient is

$$\frac{d^2u}{dx^2}=6x-6.$$

Substituting 4 for x in the second differential coefficient, the result is $+18$, which, being positive, indicates a minimum; sub-

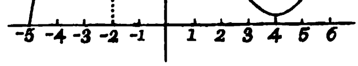
stituting -2 for x, the result is -18, which, being negative, indicates a maximum.

Hence the proposed function has a maximum value when $x=-2$, and a minimum value when $x=4$.

This result may be illustrated by assuming a series of values for x, and computing the corresponding values of u. Thus,

If $x=-4$, $u=69$,
$x=-3$, $u=103$,
$x=-2$, $u=113$ *maximum.*
$x=-1$, $u=105$,
$x=0$, $u=85$,
$x=+1$, $u=59$,
$x=+2$, $u=33$,
$x=+3$, $u=13$,
$x=+4$, $u=5$ *minimum.*
$x=+5$, $u=15$,
$x=+6$, $u=49$.

Thus it is seen that the value of the function increases, while x increases from -4 to -2; it then decreases till $x=4$, and after that it increases again uninterruptedly, and will continue to do so till x equals infinity. This peculiarity may be illustrated by a figure.



If we assume the different values of x to represent the abscissas of a curve, and erect ordinates equal to the corresponding values of u, the curve line which passes through the extremities of all the ordinates, will be of the form represented in the annexed figure, where it is evident that the ordinates attain a maximum corresponding to the abscissa -2, and a minimum corresponding to the abscissa 4.

Ex. 2. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-18x^2+96x-20.$$

Ans. $x=4$ renders the function a maximum, and $x=8$ renders it a minimum.

Ex. 3. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^3-18x^2+105x.$$

Ans. This function has a maximum value when $x=5$, and a minimum value when $x=7$.

Illustrate these results by a figure, as in the preceding example.

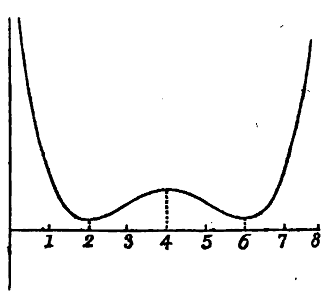
Ex. 4. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^4-16x^3+88x^2-192x+150.$$

Ans. This function has a maximum value when $x=4$, and a minimum value when $x=2$ or 6.

If we assume a series of values for x, we shall obtain the corresponding values of u as follows:

If $x=1$, $u=$ 31,
$x=2$, $u=$ 6 *minimum.*
$x=3$, $u=$ 15,
$x=4$, $u=$ 22 *maximum.*
$x=5$, $u=$ 15,
$x=6$, $u=$ 6 *minimum.*
$x=7$, $u=$ 31,
$x=8$, $u=150$.



The curve representing these values has the form represented in the annexed figure, where two minima are seen corresponding to the abscissas 2 and 6, and a maximum corresponding to the abscissa 4.

Ex. 5. Find the values of x which will render u a maximum or a minimum in the equation

$$u=x^5-25x^4+\frac{700}{3}x^3-1000x^2+1920x-1100.$$

Ans. This function has two maximum values corresponding to $x=2$ and $x=6$, and two minimum values corresponding to $x=4$ and $x=8$.

(210.) The following principles will often enable us to abridge the process of finding maxima and minima:

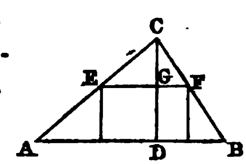
1. If the proposed function is multiplied or divided by a constant quantity, the same values of x which render the function a maximum or a minimum, will also render the product or quotient a maximum or minimum; hence *a constant factor may be omitted.*

2. Whatever value of x renders a function a maximum or a minimum, must obviously render its square, cube, and every other power a maximum or a minimum; and hence, *if a function is under a radical, the radical may be omitted.*

In the solution of problems of maxima and minima, we must obtain an algebraic expression for the quantity whose maximum or minimum state is required, find its first differential coefficient, and place this equal to zero; from which equation the value of the variable x, corresponding to a maximum or a minimum, will be obtained.

(211.) The following examples will illustrate these principles:

Ex. 1. It is required to find the maximum rectangle which can be inscribed in a given triangle.



Let b represent the base of the triangle ABC, h its altitude, and x the altitude of the inscribed rectangle. Then, by similar triangles, we have

$$CD : CG :: AB : EF,$$

or $$h : h-x :: b : EF.$$

Hence $$EF=\frac{b}{h}(h-x).$$

Therefore the area of the rectangle is equal to EF×GD,

or $$\frac{b}{h}(hx-x^2),$$

which is to be a maximum.

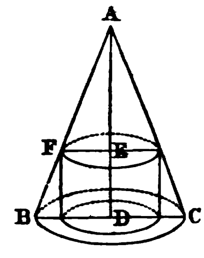
But since $\frac{b}{h}$ is a constant, the quantity $hx-x^2$ will also be a maximum, Art. 210.

Hence $$\frac{du}{dx}=h-2x=0,$$

or $$x=\frac{h}{2}.$$

Hence the altitude of the rectangle is equal to half the altitude of the triangle.

Ex. 2. What is the altitude of a cylinder inscribed in a given right cone when the solidity of the cylinder is a maximum?



Let a represent the height of the cone, b the radius of its base, and x the altitude of the inscribed cylinder. Then, by similar triangles, we have

$$AD : BD :: AE : EF,$$

or $$a : b :: a-x : EF.$$

Hence $$EF=\frac{b}{a}(a-x).$$

Now the area of a circle whose radius is R is πR^2 (Geom., Prop. XIII., Cor. 3, B. VI.). Hence the area of a circle whose radius is EF is

$$\frac{\pi b^2}{a^2}(a-x)^2.$$

Multiplying this surface by DE, the height of the cylinder, we obtain its solidity,

$$\frac{\pi b^2}{a^2}x(a-x)^2,$$

which is to be a maximum.

Neglecting the constant factor $\frac{\pi b^2}{a^2}$, we have

$$u=x(a-x)^2=a^2x-2ax^2+x^3, \text{ a maximum.}$$

Differentiating, we have

$$\frac{du}{dx}=a^2-4ax+3x^2=0,$$

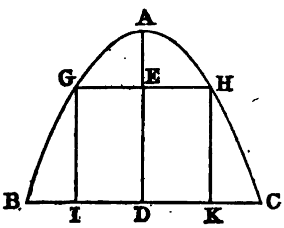
where x may equal a or $\frac{1}{3}a$.

The second differential coefficient is

$$\frac{d^2u}{dx^2}=-4a+6x.$$

The value $x=a$ reduces the second differential coefficient to a positive quantity, indicating a minimum; the value $x=\frac{1}{3}a$ reduces this coefficient to a negative quantity, indicating a maximum; that is, the height of the greatest cylinder is one third the altitude of the cone.

Ex. 3. What is the altitude of the maximum rectangle which can be inscribed in a given parabola?



Put AD$=a$ and AE$=x$; then, by the equation of the parabola $y^2=2px$, we have

$$GE^2=2px.$$

Hence GE$=\sqrt{2px}$, and GH$=2\sqrt{2px}$.

Therefore the area of GHKI is $2\sqrt{2px}(a-x)$, which is a maximum, or $\sqrt{x}(a-x)$ is a maximum.

Hence $$u=ax^{\frac{1}{2}}-x^{\frac{3}{2}},$$

And $$\frac{du}{dx}=\tfrac{1}{2}ax^{-\frac{1}{2}}-\tfrac{3}{2}x^{\frac{1}{2}}=0.$$

Hence $$\frac{a}{x^{\frac{1}{2}}}=3x^{\frac{1}{2}},$$

or $$a=3x, \text{ and } x=\tfrac{1}{3}a.$$

Consequently the altitude of the maximum rectangle is two thirds of the axis of the parabola.

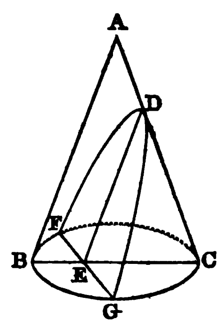
There is a parabola whose abscissa is 9, and double ordinate 16; required the sides of the greatest rectangle which can be inscribed in it. *Ans.*

Ex. 4. What is the length of the axis of the maximum parabola which can be cut from a given right cone?

Put BC$=a$, AB$=b$, and CE$=x$, then BE$=a-x$, FE$=\sqrt{ax-x^2}$, Geom., Prop. XXII., Cor., B. IV., and FG$=2\sqrt{ax-x^2}$.

By similar triangles we have

$$a:b::x:\text{DE}=\frac{bx}{a}.$$



Hence the area of the parabola, Art. 65, is

$$\frac{2}{3}\cdot\frac{bx}{a}\cdot 2\sqrt{ax-x^2}, \text{ a maximum.}$$

Hence we find $x=\frac{3}{4}a$,

and $$\text{DE}=\frac{bx}{a}=\tfrac{3}{4}b;$$

that is, the axis of the maximum parabola is three fourths the side of the cone.

Ex. 5. Divide a into two parts such that the least part, multiplied by the square of the greatest, may be a maximum.

Ans. $\dfrac{a}{3}$ and $\dfrac{2a}{3}$.

Ex. 6. Divide a into two parts such that the least, multiplied by the cube of the greatest, may be a maximum.

Let x represent the greater part, then

$$x^3(a-x) = \text{a maximum},$$

and $$x = \tfrac{3}{4}a.$$

Ex. 7. It is required to determine the dimensions of a cylindrical vessel open at top, which has the least surface with a given capacity.

Let c denote the capacity of the vessel, and x the radius of the base, the area of the base will be represented by πx^2.

Hence the height of the cylinder equals $\dfrac{c}{\pi x^2}$.

The convex surface of the cylinder is $\dfrac{c}{\pi x^2} \times 2\pi x = \dfrac{2c}{x}$.

Adding to this the area of the base, we have

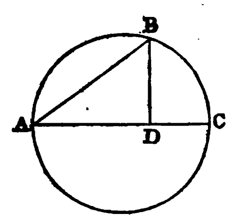
$$\frac{2c}{x} + \pi x^2, \text{ a minimum},$$

from which we obtain

$$x = \sqrt[3]{\frac{c}{\pi}}.$$

Substituting this value of x in the expression for the height, we find the height $= \sqrt[3]{\dfrac{c}{\pi}}$; that is, the altitude of the cylinder is equal to the radius of the base.

Ex. 8. Required the altitude of a cone inscribed in a given sphere, which shall render the convex surface of the cone a maximum.



Let $AC = 2a$, and $AD = x$, then

$$x : BD :: BD : 2a - x.$$

Whence $$BD = \sqrt{2ax - x^2}.$$

Also, $$x : AB :: AB : 2a.$$

Whence $$AB = \sqrt{2ax}.$$

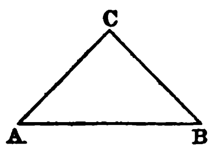
The convex surface of the cone $= \pi\sqrt{2ax - x^2}\sqrt{2ax}$,

$$= \pi\sqrt{4a^2x^2 - 2ax^3}, \text{ a maximum}.$$

Whence $$x = \tfrac{4}{3}a;$$

that is, the altitude of the cone whose convex surface is a maximum, is $\tfrac{4}{3}$ of the radius of the sphere.

Ex. 9. Required the greatest right-angled triangle which can be constructed upon a given line.

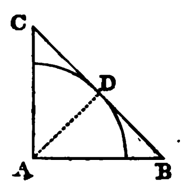


Let a represent the hypothenuse AB, and x one of the sides of the triangle, the other side will be $\sqrt{a^2-x^2}$, and the surface of the triangle will be $\frac{x}{2}\sqrt{a^2-x^2}$.

Whence $$2x^2=a^2.$$

Therefore the two sides of the required triangle are equal to each other.

Ex. 10. Required the least triangle which can be formed by the radii produced, and a tangent line to the quadrant of a given circle.



Let ABC be the required triangle, and draw AD from the right angle perpendicular to the hypothenuse. The area of the triangle ABC is equal to $\frac{1}{2}\text{AD}\times\text{BC}$, which will be a minimum when BC is a minimum, because AD is a constant quantity.

Let AD=R, and BD=x; then, Geom., Prop. XXII., B. IV.,

$$\text{BD} : \text{AD} :: \text{AD} : \text{DC},$$

or $$\text{DC}=\frac{\text{AD}^2}{\text{BD}}=\frac{\text{R}^2}{x}.$$

Hence $$\text{BC}=x+\frac{\text{R}^2}{x}.$$

Therefore x=R, and DC=R; that is, the two sides of the required triangle are equal to each other.

Ex. 11. A right-angled triangle is to contain a given area; required the base and perpendicular so that their sum may be the least possible.

Ans.

Ex. 12. Required the least square which can be inscribed in a given square.

Ans. Each angle of the required square is on the middle of a side of the given square.

Ex. 13. Required the sides of the maximum rectangle inscribed in a given circle.

Ans. Each is equal to R$\sqrt{2}$.

Ex. 14. Required the maximum cone which can be inscribed in a given sphere.

Ans.

Ex. 15. It is required to determine the dimensions of a cylinder which shall contain a cubic foot, and have the least possible surface, including both ends.

Ans.

Ex. 16. A carpenter has a tapering tree of valuable wood, the diameter of the larger end being three feet, and that of the smaller end a foot and a half, and the length 20 feet; and he wishes to cut the largest possible cylinder out of it; required the length and diameter of the cylinder?

Ans.

Ex. 17. A cabinet-maker has a mahogany board, the breadth at one end being 4 feet, and at the other 2, and its length 10 feet; and he wishes to cut the largest possible rectangular table out of it. At what distance from the narrow end must it be cut?

Ans.

SECTION IV.

OF TRANSCENDENTAL FUNCTIONS.

(212.) An *algebraic* function is one in which the relation between the function and variable can be expressed by the ordinary operations of algebra, as in the functions hitherto considered.

A *transcendental* function is one in which the relation between the function and variable can not be expressed by the ordinary operations of algebra; as,

$$u=\sin. x, \quad u=\text{tang}. x, \quad u=\sec. x, \text{ etc.},$$

which are called *circular* functions,

or
$$u=\log. x, \quad u=a^x,$$

which are called *logarithmic* or *exponential* functions.

Proposition I.—Theorem.

(213.) *The differential of a constant quantity raised to a power denoted by a variable exponent, is equal to the power multiplied by the Naperian logarithm of the root, into the differential of the exponent.*

Let us take the exponential function

$$u=a^x,$$

and give to x an increment h, we shall have

$$u'=a^{x+h}=a^x a^h.$$

Therefore
$$u'-u=a^x a^h-a^x=a^x(a^h-1). \qquad (1)$$

In order that a^h may be developed into a series by the binomial theorem, let us assume

$$a=1+b,$$

we shall then have

$$a^h=(1+b)^h=1+h\frac{b}{1}+h(h-1)\frac{b^2}{1.2}+h(h-1)(h-2)\frac{b^3}{1.2.3}+, \text{ etc.}$$

The second member of this equation consists of a series of

terms involving the first power of h, together with terms involving the higher powers. It may, therefore, be written

$a^h=(1+b)^h=1+\left(\frac{b}{1}-\frac{b^2}{2}+\frac{b^3}{3}-, \text{ etc.}\right)h+$ terms involving higher powers of h.

Let us put k for the expression

$$\frac{b}{1}-\frac{b^2}{2}+\frac{b^3}{3}-, \text{ etc.},$$

and we shall have

$$a^h-1=kh+ \text{ terms involving } h^2, \text{ etc.}$$

Substituting this value in equation (1), it becomes

$$u'-u=a^x kh+ \text{ terms in } h^2, \text{ etc.},$$

or

$$\frac{u'-u}{h}=a^x k+ \text{ terms involving } h, \text{ etc.},$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174; the result will be the differential coefficient. Hence

$$\frac{du}{dx}=\frac{da^x}{dx}=a^x k.$$

The symbol k represents a constant quantity depending on a, and its value may be found by Maclaurin's theorem. We have found

$$\frac{da^x}{dx}=a^x k.$$

Hence

$$d\left(\frac{da^x}{dx}\right)=da^x k=a^x k^2 dx.$$

Therefore

$$\frac{d^2a^x}{dx^2}=a^x k^2.$$

Also,

$$\frac{d^3a^x}{dx^3}=a^x k^3, \text{ etc.}$$

If in the function $u=a^x$, and the successive differential coefficients thus found, we make $x=0$, we shall obtain

$$(u)=1,\ \left(\frac{du}{dx}\right)=k,\ \left(\frac{d^2u}{dx^2}\right)=k^2,\ \frac{d^3u}{dx^3}=k^3, \text{ etc.}$$

Hence, by substitution, Art. 191,

$$a^x=1+\frac{kx}{1}+\frac{k^2x^2}{1.2}+\frac{k^3x^3}{1.2.3}+, \text{ etc.}$$

If we now make $x=\frac{1}{k}$, we shall have

$$a^{\frac{1}{k}}=1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\text{, etc.}$$

The sum of this series is 2.718282, which is the base of the Naperian system of logarithms. Representing it by e, we shall have

$$a^{\frac{1}{k}}=e,$$

or

$$a\ =e^{k}.$$

Hence *the constant quantity* k *is the Naperian logarithm of* a, which we will denote by log.′ a. Hence

$$\frac{da^x}{dx}=a^x \ \log.' \, a,$$

or

$$da^x=a^x.\ \log.' \, a.dx.$$

PROPOSITION II.—THEOREM.

(214.) *The differential of the logarithm of a quantity is equal to the modulus of the system, into the differential of the quantity divided by the quantity itself.*

According to the preceding Proposition,

$$da^x=a^x \ \log.' \, a.dx.$$

Put $u=a^x$, and we find

$$dx=\frac{du}{u \log.' \, a}.$$

If a be the base of a system of logarithms, then x is the logarithm of u in that system, and $\frac{1}{\log.' \, a}$, Algebra, Art. 349, is the modulus of the system, which we will represent by M.

Hence $\qquad dx=d.\log.\ u=\text{M}.\frac{du}{u}.$

(215.) *Cor.* For the Naperian system of logarithms M=1, and the preceding expression becomes

$$d.\log.' \, u=\frac{du}{u};$$

that is, *the differential of the Naperian logarithm of a quantity, is equal to the differential of the quantity divided by the quantity itself.*

Ex. 1. Required the differential of the common logarithm of 4825.

From Art. 214 it appears that if we divide the modulus of the system of logarithms by 4825 (regarding its differential as unity), we shall obtain the differential of the logarithm of 4825. This division may be performed arithmetically, or by the use of logarithms, thus:

The modulus of the common system is .434294, whose logarithm is	9.637784.
The logarithm of 4825 is	3.683497.
The difference is	5.954287.

The number corresponding to this logarithm is

.000090,

which is the difference between the logarithm of 4825 and that of 4826, as is seen in the Table of Logarithms, p. 11.

Ex. 2. Required the differential of the common logarithm of 9651. *Ans.* .000045.

Ex. 3. Required the differential of the common logarithm of 5791. *Ans.* .000075.

Ex. 4. Required the differential of the common logarithm of 3810. *Ans.* .000114.

(216.) By combining the preceding theorems with those before given, we may differentiate complex, exponential, and logarithmic functions.

In the following examples, Naperian logarithms are supposed to be employed. If the logarithms are taken in any other system, we have merely to multiply the results by the modulus of that system.

Ex. 1. Differentiate the function $u=\log.\frac{a+x}{a-x}$, by the rule for fractions and that for logarithms.

$$Ans.\ \frac{2adx}{a^2-x^2}.$$

Ex. 2. Differentiate the function $u=\log.\frac{x}{\sqrt{a^2+x^2}}$.

$$Ans.\ \frac{a^2dx}{x(a^2+x^2)}.$$

Ex. 3. Differentiate the function $u=(a^x+1)^2$.

Ans. $2a^x(a^x+1)\log.a.dx$.

Ex. 4. Differentiate the function $u=\frac{a^x-1}{a^x+1}$, by the rule for fractions and that for exponential functions, and find the numerical rate of increase of the function when $a=10$, and x becomes 2.

$$Ans.\ \frac{2a^x\log.a.dx}{(a^x+1)^2}=0.045dx.$$

The Naperian logarithm of 10 is 2.302. See Alg., Art. 348.

Ex. 5. Differentiate the function

$$u=y^x,$$

in which y and x are both variables.

If we take the logarithm of each member of this equation, we shall have

$$\log.u=x\log.y.$$

Hence, Art. 215, $\quad \frac{du}{u}=\log.ydx+x\frac{dy}{y},$

or $\quad du=u\log.ydx+ux\frac{dy}{y};$

or, by substituting for u its value, we have

$$du=dy^x=y^x\log.ydx+xy^{x-1}dy,$$

which is evidently the sum of the differentials which arise by differentiating, first under the supposition that x *varies and* y *remains constant, and then under the supposition that* y *varies and* x *remains constant.*

Ex. 6. Differentiate the function $u=\frac{a^x}{x^x}$ or $\left(\frac{a}{x}\right)^x$.

$$Ans.\ \left(\frac{a}{x}\right)^x(\log.\frac{a}{x}-1)dx.$$

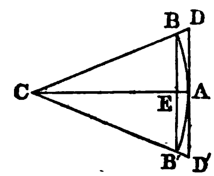
DIFFERENTIATION OF CIRCULAR FUNCTIONS.

PROPOSITION III.—THEOREM.

(217.) *An arc of a circle not exceeding a quadrant, is greater than its sine, and less than its tangent.*

Let AB be an arc of a circle whose sine is BE and tangent AD. Take AB′ equal to AB, and draw the sine B′E, and the tangent D′A.

The chord BB′, being a straight line, is shorter than the arc BAB′; therefore the sine BE, half of BB′, is less than the arc BA, half of the arc BAB′. Therefore the sine is less than the arc.



Again, the area of the sector ABC is measured by

$$\tfrac{1}{2}\text{AC}\times\text{arc AB}.$$

Also, the area of the triangle ADC is measured by

$$\tfrac{1}{2}\text{AC}\times\text{AD}.$$

But the sector ABC is less than the triangle ADC, being contained within it; hence

$$\tfrac{1}{2}\text{AC}\times\text{arc AB}<\tfrac{1}{2}\text{AC}\times\text{AD}.$$

Consequently $\qquad \text{arc AB}<\text{AD};$

that is, the arc is less than the tangent.

(218.) *Cor.* 1. *The limit of the ratio of the sine to the a. unity;* for when the arc h represented by AB becomes zero, the ratio of the sine to the tangent is unity. Since, by Trig., Art. 28,

$$\frac{\text{sin.}}{\text{tang.}}=\frac{\text{cos.}}{\text{R}}.$$

But the cosine of 0 is equal to radius; hence, when $h=0$,

$$\frac{\text{sin. } h}{\text{tang. } h}=1,$$

and since the arc is always comprised between the sine and the tangent, we must have, when we pass to the limit,

$$\frac{\text{sin. } h}{h}=1.$$

Cor. 2. Since the chord of any finite arc is less than the arc, but greater than the sine, and we have found that the limiting ratio of the sine to the arc is unity, *the limiting ratio of the chord to the arc is unity.*

PROPOSITION IV.—THEOREM.

(219.) *The differential of the sine of an arc is equal to the cosine of the arc, into the differential of the arc, divided by radius.*

Let $\qquad u=\text{sin. } x.$

If we increase x by h, then

$$u'=\sin.\,(x+h),$$

and
$$u'-u=\sin.\,(x+h)-\sin.\,x. \qquad (1)$$

But by Trig., Art. 75,

$$\sin.\,A-\sin.\,B=\frac{2}{R}\sin.\,\tfrac{1}{2}(A-B)\cos.\,\tfrac{1}{2}(A+B). \qquad (2)$$

Put $A=x+h$, and $B=x$, equation (2) becomes

$$\sin.\,(x+h)-\sin.\,x=\frac{2}{R}\sin.\,\tfrac{1}{2}h\cos.\,(x+\tfrac{1}{2}h).$$

Hence equation (1) becomes

$$u'-u=\frac{2}{R}\sin.\,\tfrac{1}{2}h\cos.\,(x+\tfrac{1}{2}h).$$

Dividing both members by h, and both terms of the fraction in the second member by 2, we have

$$\frac{u'-u}{h}=\frac{\sin.\,\frac{1}{2}h}{\frac{1}{2}h}\times\frac{\cos.\,(x+\frac{1}{2}h)}{R},$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174. But in this case, according to the last proposition, Cor. 1,

$$\frac{\sin.\,\frac{1}{2}h}{\frac{1}{2}h}=1.$$

Hence
$$\frac{du}{dx}=\frac{\cos.\,x}{R},$$

or
$$du=d\sin.\,x=\frac{\cos.\,xdx}{R}.$$

Ex. 1. Required the differential of the sine of 30°, the differences being taken for single minutes.

The differential of x, which is 1′, must be taken in parts of radius, which, on p. 150 of the Tables, is found to be .0002909.

Its logarithm is	6.463726.
The logarithm cosine of 30° is	9.937531.
Their sum is	6.401257

The natural number corresponding to this logarithm is

0.000252,

which, on p. 122 of the Tables, is seen to be the difference between the natural sine of 30° and the sine of 30° 1′.

Ex. 2. Required the differential of the sine of 10° 31′.

Ans. 0.000286

Ex. 3. Required the differential of the sine of 60° 46′.

Ans. 0.000142.

Ex. 4. Required the differential of the sine of 80° 41′.

Ans. 0.000047.

PROPOSITION V.—THEOREM.

(220.) *The differential of the cosine of an arc is negative, and is equal to the sine of the arc into the differential of the arc, divided by radius.*

Let $u = \cos. x.$

Then $du = d\cos. x = d\sin. (90° - x).$ (1)

But, by the last Proposition,

$$d\sin. (90° - x) = \frac{\cos. (90° - x)d(90° - x)}{R}.$$

But $\cos. (90° - x) = \sin. x,$

and $d(90° - x) = -dx.$

Hence, by substitution, equation (1) becomes

$$d\cos. x = -\frac{\sin. xdx}{R}.$$

Cor. Since the versed sine of an arc is equal to radius, minus the cosine, we have

$$d.\,\text{versed}\sin. x = d(R - \cos. x) = \frac{\sin. xdx}{R}.$$

Ex. 1. Required the differential of the cosine of 65° 10′.

Ans. −0.000264.

Ex. 2. Required the differential of the cosine of 5° 31′.

Ans. −0.000028.

As the arc increases, its cosine diminishes; hence its differential is negative.

PROPOSITION VI.—THEOREM.

(221.) *The differential of the tangent of an arc is equal to the square of radius, into the differential of the arc, divided by the square of the cosine of the arc.*

Let $u = \text{tang. } x.$

Since $\text{tang. } x = \frac{R\sin. x}{\cos. x}$, we have, Art. 184,

$$d.\text{tang. } x=\frac{\text{R cos. } xd \text{ sin. } x-\text{R sin. } xd \text{ cos. } x}{\cos.^2 x},$$

$$=\frac{(\cos.^2 x+\sin.^2 x)dx}{\cos.^2 x}.$$

But $\cos.^2 x+\sin.^2 x=\text{R}^2.$

Hence $d.\text{tang. } x=\frac{\text{R}^2dx}{\cos.^2 x}.$

Ex. 1. Required the differential of the tangent of 45°.
Ans. 0.00058.

Ex. 2. Required the differential of the tangent of 84° 14′.
Ans. 0.02890.

PROPOSITION VII.—THEOREM.

(222.) *The differential of the cotangent of an arc is negative, and is equal to the square of radius into the differential of the arc, divided by the square of the sine of the arc.*

Let $u=\cot.\ x.$

$$du=d\cot.\ x=d\text{ tang. }(90°-x). \qquad (1)$$

But by the last Proposition,

$$d\text{ tang. }(90°-x)=\frac{\text{R}^2d(90°-x)}{\cos.^2(90°-x)}.$$

Also, $d(90°-x)=-dx,$

and $\cos.^2(90°-x)=\sin.^2 x.$

Hence, by substitution, equation (1) becomes

$$d.\cot.\ x=-\frac{\text{R}^2dx}{\sin.^2 x}.$$

Ex. 1. Required the differential of the cotangent of 35° 6′.
Ans. 0.00088.

Ex. 2. Required the differential of the cotangent of 9° 35′.
Ans. 0.01048.

The preceding are the differentials of the *natural* sines, tangents, etc. The differentials of the *logarithmic* sines and tangents may be found by combining Proposition II. with the preceding.

PROPOSITION VIII.—THEOREM.

(223.) *The differential of the logarithmic sine of an arc, is equal to the modulus of the system into the differential of the arc, divided by the tangent of the arc.*

By Proposition II.,

$$d\log.\sin.x=\frac{\mathrm{M}d\sin.x}{\sin.x}=\frac{\mathrm{M}\cos.xdx}{\mathrm{R}\sin.x}.$$

But Trig., Art. 28, $\tan g.x=\frac{\mathrm{R}\sin.x}{\cos.x}.$

Hence $$d\log.\sin.x=\frac{\mathrm{M}dx}{\tan g.x}.$$

Ex. 1. Required the differential of the logarithmic sine of 10′ 30″, the difference being taken for single seconds.

The differential of x, which is 1″, must be taken in parts of radius, which, on p. 150 of Tables, is found to be .00000485.

Its logarithm is	4.685575.
The modulus M log. . .	9.637784.
Mdx	4.323359.
tang. 10′ 30″	7.484917.
0.000689 =	6.838442.

Therefore 0.000689 is the difference between the logarithmic sine of 10′ 30″ and 10′ 31″, which corresponds with p. 22 of the Tables.

Ex. 2. Required the differential of the log. sine of 4° 28′.

Ans. 0.000027.

PROPOSITION IX.—THEOREM.

(224.) *The differential of the logarithmic cosine of an arc is negative, and is equal to the modulus of the system into the tangent of the arc into the differential of the arc, divided by the square of radius.*

By Proposition II.,

$$d\log.\cos.x=\frac{\mathrm{M}d\cos.x}{\cos.x}=-\frac{\mathrm{M}\sin.xdx}{\mathrm{R}\cos.x}.$$

But Trig., Art. 28, $\frac{\sin.x}{\cos.x}=\frac{\tan g.x}{\mathrm{R}}.$

Hence $$d\log.\cos.x=-\frac{\mathrm{M}\tan g.xdx}{\mathrm{R}^2}.$$

Ex. 1. Required the differential of the log. cosine of 67° 30′.
Ans. 0.000005.

Ex. 2. Required the differential of the log. cosine of 89° 30′.
Ans. 0.000245.

PROPOSITION X.—THEOREM.

(225.) *The differential of the logarithmic tangent or cotangent of an arc, is equal to the modulus of the system into the tangent of the arc into the differential of the arc, divided by the square of the sine of the arc.*

By Proposition II.,

$$d \log. \operatorname{tang.} x = \frac{Md \operatorname{tang.} x}{\operatorname{tang.} x} = \frac{MR^2 dx}{\cos.^2 x \operatorname{tang.} x}.$$

But Trig., Art. 28,

$$\frac{R}{\cos. x} = \frac{\operatorname{tang.} x}{\sin. x}, \text{ or } \frac{R^2}{\cos.^2 x} = \frac{\operatorname{tang.}^2 x}{\sin.^2 x}.$$

Hence
$$d \log. \operatorname{tang.} x = \frac{M \operatorname{tang.} x dx}{\sin.^2 x}.$$

Again,
$$d \log. \cot. x = \frac{Md \cot. x}{\cot. x} = -\frac{MR^2 dx}{\sin.^2 x \cot. x}.$$

But
$$\frac{R^2}{\cot. x} = \operatorname{tang.} x.$$

Hence
$$d \log. \cot. x = -\frac{M \operatorname{tang.} x dx}{\sin.^2 x}.$$

Hence the differentials of the log. tangent and cotangent of any arc, differ only in sign.

It may also be easily proved that the differential of the logarithmic tangent is equal to the arithmetical sum of the differentials of the sine and cosine of the same arc.

Ex. 1. Required the differential of the log. tangent of 10′ 30″.
Ans. 0.000689.

Ex. 2. Required the differential of the log. tangent of 10° 16′.
Ans. 0.000012.

Ex. 3. Required the differential of the log. tangent of 89° 4′ 30″.
Ans. 0.000130.

(226.) We have found the differentials of the sine, cosine, etc., in terms of the arc as an independent variable. It is sometimes more convenient to regard the arc as the function,

and the sine, cosine, etc., as the variable. Let us represent any arc by z, and let us put

$$y=\sin. z.$$

By Art. 219, we have

$$dy=\frac{\cos. z dz}{R};$$

whence

$$dz=\frac{Rdy}{\cos. z}. \qquad (1)$$

But

$$\cos.^2 z+\sin.^2 z=R^2;$$

whence

$$\cos. z=\sqrt{R^2-\sin.^2 z}=\sqrt{R^2-y^2}.$$

Substituting this value in equation (1), we obtain

$$dz=\frac{Rdy}{\sqrt{R^2-y^2}},$$

which is the differential of an arc, its sine being regarded as the independent variable.

Let us put

$$y'=\cos. z.$$

By Art. 220, we have

$$dy'=-\frac{\sin. z dz}{R};$$

whence

$$dz=-\frac{Rdy'}{\sin. z}.$$

But

$$\sin. z=\sqrt{R^2-\cos.^2 z}=\sqrt{R^2-y'^2}.$$

Hence

$$dz=-\frac{Rdy'}{\sqrt{R^2-y'^2}},$$

which is the differential of an arc, its cosine being regarded as the independent variable.

Let us put

$$x=\text{versed sine } z.$$

By Art. 220, Cor., we have

$$dx=\frac{\sin. z dz}{R};$$

whence

$$dz=\frac{Rdx}{\sin. z}.$$

But $\sin. z=\sqrt{(2R-x)x}$, Geom., Prop. XXII., Cor., B. IV., and, consequently,

$$dz=\frac{Rdx}{\sqrt{2Rx-x^2}},$$

which is the differential of an arc, its versed sine being regarded as the independent variable.

Let us put $t=\text{tang. } z.$

By Art. 221, we have

$$dt=\frac{R^2dz}{\cos.^2 z};$$

hence $$dz=\frac{\cos.^2 z dt}{R^2}.$$

But (Trig., Art. 28) $\frac{\cos. z}{R}=\frac{R}{\sec. z}.$

Hence $$\frac{\cos.^2 z}{R^2}=\frac{R^2}{\sec.^2 z}=\frac{R^2}{R^2+\text{tang.}^2 z}=\frac{R^2}{R^2+t^2}.$$

Therefore $$dz=\frac{R^2dt}{R^2+t^2},$$

which is the differential of an arc, its tangent being regarded as the independent variable.

(227.) If we make R equal to unity, these formulas become

$$1. \quad dz=\frac{dy}{\sqrt{1-y^2}},$$

where y represents the sine of the arc z.

$$2. \quad dz=\frac{-dy'}{\sqrt{1-y'^2}},$$

where y' represents the cosine of the arc z.

$$3. \quad dz=\frac{dx}{\sqrt{2x-x^2}},$$

where x represents the versed sine of the arc z.

$$4. \quad dz=\frac{dt}{1+t^2},$$

where t represents the tangent of the arc z.

Ex. 1. If two bodies start together from the extremity of the diameter of a circle, the one moving uniformly along the diameter at the rate of 10 feet per second, and the other in the circumference with a variable velocity so as to keep it always perpendicularly above the former; what is its velocity in the circumference when passing the sixtieth degree from the starting point, supposing the diameter of the circle to be 50 feet?

Ans. $\frac{20}{\sqrt{3}}$ feet per second.

Ex. 2. If two bodies start together from the extremity of the

diameter of a circle, the one moving uniformly along the tangent at the rate of 10 feet per second, and the other in the circumference with a variable velocity, so as to be always in the straight line joining the first body with the center of the circle; what is its velocity when passing the forty-fifth degree from the starting point, the diameter of the circle being 50 feet?

Ans. 5 feet per second.

(228.) Maclaurin's theorem enables us to develop the sine of x, cosine of x, etc., in terms of the ascending powers of x.

Ex. 1. It is required to develop sin. x into a series.

Let $u=\sin. x$, and $R=$unity.

By Art. 219,
$$\frac{du}{dx}=\cos. x,\quad \frac{d^2u}{dx^2}=-\sin. x,$$
$$\frac{d^3u}{dx^3}=-\cos. x,\quad \frac{d^4u}{dx^4}=+\sin. x, \text{ etc.}$$

If now we make $x=0$, we shall have
$$(u)=0,\quad \left(\frac{du}{dx}\right)=1,\quad \left(\frac{d^2u}{dx^2}\right)=0,$$
$$\left(\frac{d^3u}{dx^3}\right)=-1,\quad \left(\frac{d^4u}{dx^4}\right)=0, \text{ etc.}$$

Therefore
$$\sin. x=x-\frac{x^3}{2.3}+\frac{x^5}{2.3.4.5}-, \text{ etc.}$$

Ex. 2. It is required to develop cos. x into a series.

Let
$$u=\cos. x.$$

By Art. 220,
$$\frac{du}{dx}=-\sin. x,\quad \frac{d^2u}{dx^2}=-\cos. x,$$
$$\frac{d^3u}{dx^3}=\sin. x,\quad \frac{d^4u}{dx^4}=\cos. x, \text{ etc.}$$

If now we make $x=0$, we shall have
$$(u)=1,\quad \left(\frac{du}{dx}\right)=0,\quad \left(\frac{d^2u}{dx^2}\right)=-1,$$
$$\left(\frac{d^3u}{dx^3}\right)=0,\quad \left(\frac{d^4u}{dx^4}\right)=1.$$

Therefore
$$\cos. x=1-\frac{x^2}{2}+\frac{x^4}{2.3.4}-, \text{ etc.}$$

These series for small values of x converge rapidly, and are very convenient for computing a table of natural sines and cosines.

SECTION V.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE THEORY OF CURVES.

(229.) If we differentiate the equation of a line, we shall obtain a new equation which expresses the relation between the differentials of the co-ordinates of the line. This equation is called *the differential equation of the line.*

If, for example, we take the equation of a straight line,

$$y=ax+b. \qquad (1)$$

and differentiate it, we find

$$\frac{dy}{dx}=a, \qquad (2)$$

a result which is the same for all values of b.

Differentiating equation (2), we obtain

$$\frac{d^2y}{dx^2}=0. \qquad (3)$$

This last equation is entirely independent of the values of a and b, and is equally applicable to every straight line which can be drawn in the plane of the co-ordinate axes. It is called *the differential equation of lines of the first order.*

(230.) If we take the equation of the circle

$$x^2+y^2=R^2, \qquad (1)$$

and differentiate it, we obtain

$$2xdx+2ydy=0,$$

or

$$\frac{dy}{dx}=-\frac{x}{y}. \qquad (2)$$

Equation (2) is independent of the value of the radius R, and hence it belongs equally to every circle referred to the same co-ordinate axes.

If we take the equation of the parabola

$$y^2=2px, \qquad (1)$$

and differentiate it, we find

$$2ydy=2pdx;$$

whence $$\frac{dy}{dx}=\frac{p}{y}. \qquad (2)$$

But from equation (1), $p=\frac{y^2}{2x}$.

Hence equation (2) becomes
$$\frac{dy}{dx}=\frac{y}{2x}. \qquad (3)$$

This equation is independent of the value of the parameter $2p$, and hence it belongs equally to every parabola referred to the same co-ordinate axes.

(231.) If we take the general equation of lines of the second order, which is, Art. 132,
$$y^2=mx+nx^2, \qquad (1)$$
and differentiate it, we obtain
$$2ydy=mdx+2nxdx. \qquad (2)$$

Differentiating again, regarding dx as constant, we obtain
$$2dy^2+2yd^2y=2ndx^2,$$
or, dividing by 2, $$dy^2+yd^2y=ndx^2. \qquad (3)$$

Eliminating m and n from equations (1), (2), and (3), we obtain
$$y^2dx^2+x^2dy^2+yx^2d^2y-2xydxdy=0,$$
which is *the general differential equation of lines of the second order.*

Hence we see that an equation may be freed of its constants by successive differentiations; and for this purpose it is necessary to differentiate it as many times as there are constants to be eliminated. The differential equations thus obtained, together with the given equation, make one more than the number of constants to be eliminated, and hence a new equation may be derived which will be freed from these constants.

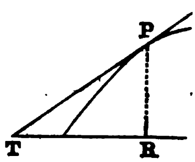
The differential equation which is obtained after the constants are eliminated, belongs to a *species* of lines, one of which is represented by the given equation.

(232.) We have seen, Art. 201, that the tangent of the angle which a tangent line at any point of a curve makes with the axis of abscissas, is equal to the first differential coefficient of the ordinate of the curve. We are enabled from this principle to deduce general expressions for the tangent and subtangent, normal and subnormal of any curve.

PROPOSITION I.—THEOREM.

(233.) *The length of the subtangent to any point of a curve referred to rectangular co-ordinates, is equal to the ordinate multiplied by the differential coefficient of the abscissa.*

In the right-angled triangle PTR, we have, Trig., Art. 42,



$$1 : \mathrm{TR} :: \text{tang. T} : \mathrm{PR};$$

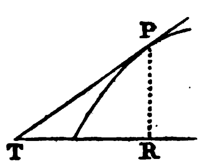
that is, $$1 : \mathrm{TR} :: \frac{dy}{dx} : y.$$

Hence the subtangent $\mathrm{TR}=y\dfrac{dx}{dy}$.

PROPOSITION II.—THEOREM.

(234.) *The length of the tangent to any point of a curve referred to rectangular co-ordinates, is equal to the square root of the sum of the squares of the ordinate and subtangent.*

In the right-angled triangle PTR,



$$\mathrm{TP}^2=\mathrm{PR}^2+\mathrm{TR}^2;$$

that is, $$\mathrm{TP}^2=y^2+y^2\frac{dx^2}{dy^2}.$$

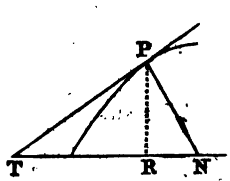
Hence the tangent

$$\mathrm{TP}=\sqrt{y^2+y^2\frac{dx^2}{dy^2}}=y\sqrt{1+\frac{dx^2}{dy^2}}.$$

PROPOSITION III.—THEOREM.

(235.) *The length of the subnormal to any point of a curve, is equal to the ordinate multiplied by the differential coefficient of the ordinate.*

In the right-angled triangle PRN, we have the proportion



$$1 : \mathrm{PR} :: \text{tang. RPN} : \mathrm{RN}.$$

But the angle RPN is equal to PTR;

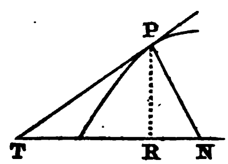
hence $$1 : \mathrm{PR} :: \text{tang. PTR} : \mathrm{RN};$$

that is, $$1 : y :: \frac{dy}{dx} : \mathrm{RN}.$$

Hence the subnormal $\mathrm{RN}=y\dfrac{dy}{dx}$.

PROPOSITION IV.—THEOREM.

(236.) *The length of the normal to any point of a curve, is equal to the square root of the sum of the squares of the ordinate and subnormal.*



In the right-angled triangle PRN,

$$PN^2=PR^2+RN^2;$$

that is, $$PN^2=y^2+\frac{y^2dy^2}{dx^2}.$$

Hence the normal $$PN=\sqrt{y^2+\frac{y^2dy^2}{dx^2}},$$

$$=y\sqrt{1+\frac{dy^2}{dx^2}}.$$

(237.) To apply these formulas to a particular curve, we must substitute in each of them the value of $\frac{dx}{dy}$ or $\frac{dy}{dx}$, obtained by differentiating the equation of the curve. The results obtained will be true for all points of the curve. If the values are required for a given point of the curve, we must substitute in these results for x and y the co-ordinates of the given point.

Let it be required to apply these formulas to lines of the second order whose general equation is

$$y^2=mx+nx^2.$$

Differentiating, we have

$$\frac{dy}{dx}=\frac{m+2nx}{2y}=\frac{m+2nx}{2\sqrt{mx+nx^2}}.$$

Substituting this value in the preceding formulas, we find

The subtangent $$=y\frac{dx}{dy}=\frac{2(mx+nx^2)}{m+2nx}.$$

The tangent $$=\sqrt{y^2+\frac{y^2dx^2}{dy^2}}=\sqrt{mx+nx^2+4\left(\frac{mx+nx^2}{m+2nx}\right)^2}.$$

The subnormal $$=y\frac{dy}{dx}=\frac{m+2nx}{2}.$$

The normal $$=\sqrt{y^2+\frac{y^2dy^2}{dx^2}}=\sqrt{mx+nx^2+\tfrac{1}{4}(m+2nx)^2}.$$

(238.) By attributing proper values to m and n, the above formulas will be applicable to each of the conic sections. For the parabola $n=0$, and these expressions become

the subtangent $=2x$, which corresponds with Art. 53;

the tangent $=\sqrt{mx+4x^2}$;

the subnormal $=\frac{m}{2}$ which corresponds with Art. 56;

the normal $=\sqrt{mx+\frac{m^2}{4}}$.

(239.) In the case of the ellipse, these expressions assume a simpler form when the origin of co-ordinates is placed at the center. The equation then becomes

$$A^2y^2+B^2x^2=A^2B^2, \tag{1}$$

whence, by differentiating, we obtain

$$A^2ydy+B^2xdx=0,$$

or

$$\frac{dx}{dy}=-\frac{A^2y}{B^2x}.$$

Hence, from Art. 233, we find the subtangent of the ellipse equals

$$-\frac{A^2y^2}{B^2x}. \tag{2}$$

But from equation (1), we have

$$\frac{A^2y^2}{B^2}=A^2-x^2.$$

Hence from equation (2), we find the subtangent of the ellipse equals

$$-\frac{A^2-x^2}{x},$$

which corresponds with Art. 78, Cor. 2.

Also, from Art. 235, we obtain the subnormal of the ellipse equals

$$y\frac{dy}{dx}=-\frac{B^2x}{A^2},$$

which corresponds with Art. 80, Cor. 1.

(240.) In the case of the circle A and B become equal, and we find

the subtangent $=-\frac{y^2}{x}$,

the tangent $=\sqrt{y^2+\frac{y^4}{x^2}}=\sqrt{\frac{R^2y^2}{x^2}}=\frac{Ry}{x}$.

the subnormal $=-x$,

the normal $=\sqrt{y^2+x^2}=R$,

which results agree with well-known principles of Geometry

Ex. 1. If the parameter of a parabola be 4 inches and the abscissa 9 inches, required the length of the ordinate and sub tangent. *Ans.*

Ex. 2. If the major axis of an ellipse be 30 inches and the minor axis 16 inches, required the length of the subtangent corresponding to an abscissa of 10 inches measured from the center. *Ans.*

Ex. 3. If the major axis of an ellipse be 6 inches and the minor axis 4 inches, required the length of the subnormal corresponding to an abscissa of 2 inches measured from the center. *Ans.*

Ex. 4. If the diameter of a circle be 10 feet, what is the length of the tangent and subtangent corresponding to an abscissa of 3 feet measured from the center? *Ans.*

(241.) Let it be required to find the value of the subtangent of the logarithmic curve.

If we differentiate the equation

$$x=\log. y, \text{ Art. } 141,$$

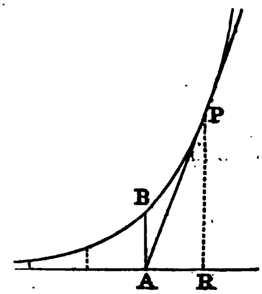
and represent the modulus of the system of logarithms by M, we obtain, Art. 214,

$$dx=\frac{Mdy}{y},$$

or

$$y\frac{dx}{dy}=M.$$

But $y\frac{dx}{dy}$ is the expression for the subtangent, Art. 233; hence *the subtangent of the logarithmic curve is constant, and equal to the modulus of the system in which the logarithms are taken.*



In the Naperian system M equals unity, and hence the subtangent AR will be equal to unity or AB.

(242.) The equation of a straight line passing through a given point, Art. 18, is

$$y-y'=a(x-x'),$$

where a denotes the tangent of the angle which the line makes with the axis of abscissas.

But we have found, Art. 201, that the first differential coefficient $\frac{dy}{dx}$ is equal to the tangent of the angle which the tangent line to a curve forms with the axis of abscissas. Hence the equation of a tangent to a curve at a point whose co-ordinates are x', y', is

$$y-y'=\frac{dy'}{dx'}(x-x'). \qquad (1)$$

And since the normal is perpendicular to the tangent, the equation of the normal, Art. 25, must be

$$y-y'=-\frac{dx'}{dy'}(x-x'). \qquad (2)$$

(243.) When it is required to find the equation of the tangent line to any curve, we must differentiate the equation of the curve, and find the value of $\frac{dy'}{dx'}$, which is to be substituted in equation (1).

Let it be required to find the equation of the tangent line to a circle.

The equation of the circle is

$$x^2+y^2=R^2;$$

and, by differentiating, we find

$$\frac{dy'}{dx'}=-\frac{x'}{y'}.$$

Substituting this value in equation (1), we have for the equation of a tangent line

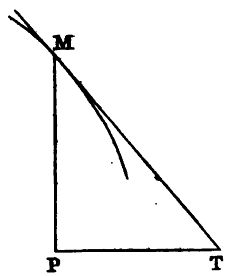
$$y-y'=-\frac{x'}{y'}(x-x').$$

Whence $$yy'+xx'=x'^2+y'^2=R^2,$$

which corresponds with Art. 40.

SUBTANGENT AND TANGENT OF POLAR CURVES.

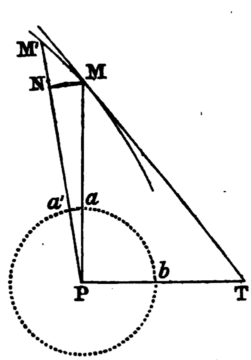
(244.) *The subtangent of a polar curve is a line drawn from the pole perpendicular to a radius vector, and limited by a tangent drawn through the extremity of the radius vector.*



Thus, if MT is a tangent to a polar curve at the point M, P the pole, and PM the radius vector, then PT, drawn perpendicular to PM, is the *subtangent.*

PROPOSITION V.—THEOREM.

(245.) *The length of the subtangent to a polar curve is equal to the square of the radius vector, multiplied by the differential coefficient of the measuring arc.*



Represent the radius vector PM by r, and the measuring arc ba by t (the radius Pa of the measuring circle being equal to unity). Suppose the arc t to receive a small increment aa', and through a' draw the radius vector PM′. With the radius PM describe the arc MN; draw the chord MN, and draw PT parallel to MN.

Now aa' is the increment of t, and M′N is the increment of r; and in order to find the differential coefficient of r (t being considered the independent variable), we must find the ratio of the increments of t and r, Art. 174, and determine the limit of this ratio by making the increment of t equal to zero.

By Geom., Prop. XIII., Cor. 1, B. VI., we have

$$1 : aa' :: \text{PM} : \text{arc MN}.$$

Hence $$aa' = \frac{\text{arc MN}}{\text{PM}}. \qquad (1)$$

Also, the similar triangles M′NM, M′PT furnish the proportion

$$\text{M'N} : \text{chord MN} :: \text{M'P} : \text{PT}.$$

Whence $$\mathrm{M'N}=\frac{\text{chord MN}\times\mathrm{M'P}}{\mathrm{PT}}. \qquad (2)$$

Consequently, from equations (1) and (2),

$$\frac{aa'}{\mathrm{M'N}}=\frac{\text{arc MN}}{\text{chord MN}}\times\frac{\mathrm{PT}}{\mathrm{PM}\times\mathrm{PM'}},$$

which is the ratio of the increments of t and r; and we must now find the limit of this ratio when the increment of t is made equal to zero.

It is evident that the ratio of the arc MN to the chord MN will be unity, Art. 218, Cor. 2; also, PT will be the subtangent, and PM′ will become equal to PM, which is represented by r.

Hence $$\frac{dt}{dr}=\frac{\mathrm{PT}}{r^2},$$

or $$\mathrm{PT}=\frac{r^2dt}{dr},$$

which is the value of the subtangent.

Cor. The tangent of the angle PMT is equal to $\frac{\mathrm{PT}}{\mathrm{PM}}$, which therefore becomes $\frac{rdt}{dr}$, which represents *the tangent of the angle which the tangent line makes with the radius vector.*

PROPOSITION VI.—THEOREM.

(246.) *The length of the tangent to a polar curve, is the square root of the sum of the squares of the subtangent and radius vector.*

For the tangent MT is equal to $\sqrt{\mathrm{MP}^2+\mathrm{PT}^2}$, which is equal to $$r\sqrt{1+\frac{dt^2}{dr^2}}.$$

(247.) It is required to apply these formulas to the spirals.

The equation of the spiral of Archimedes, Art. 148, is

$$r=\frac{t}{2\pi}.$$

Whence $$\frac{dt}{dr}=2\pi.$$

Substituting the values of r and $\frac{dt}{dr}$ in the general expression for the subtangent, Art. 245, we have

$$subtangent=\frac{t^2}{2\pi}.$$

If $t=2\pi$, that is, if the tangent be drawn at the extremity of the arc generated in one revolution, we have

the subtangent$=2\pi=$*the circumference of the measuring circle.*

If $t=2m\pi$, that is, if the tangent be drawn at the extremity of the arc generated in m revolutions, we have

$$subtangent=m \,.\, 2m\pi;$$

that is, *the subtangent after* m *revolutions, is equal to* m *times the circumference of the circle described with the radius vector of the point of contact.*

248.) The equation of the hyperbolic spiral, Art. 151, is

$$r=\frac{a}{t}.$$

Whence $$\frac{dt}{dr}=-\frac{t^2}{a}.$$

Substituting this value in the general expression for the subtangent, we have

$$subtangent=-\frac{r^2t^2}{a}=-a;$$

that is, *in the hyperbolic spiral the subtangent is constant.*

(249.) The equation of the logarithmic spiral, Art. 155, is

$$t=\log. r.$$

Whence $$dt=\frac{\mathrm{M}dr}{r},$$

and $$\frac{rdt}{dr}=\mathrm{M},$$

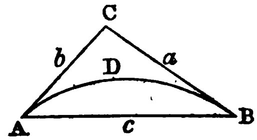
which represents the tangent of the angle which the tangent line makes with the radius vector, Prop. V., Cor.; that is, *the tangent of the angle which the tangent line makes with the radius vector is constant, and is equal to the modulus of the system of logarithms employed.*

DIFFERENTIALS OF AN ARC, AREA, SURFACE, AND SOLID OF REVOLUTION.

PROPOSITION VII.—THEOREM.

(250.) *The limit of the ratio of the chord and arc of any curve is unity.*

Let ADB be an arc of any curve, AB$=c$ the chord, and let



the tangents AC, CB be drawn at the extremities of the arc.

It is evident that the arc ADB is greater than the chord c, but less than the sum of the two tangents a and b.

By Trigonometry, Art. 49,

$$\frac{a}{c}=\frac{\sin. A}{\sin. C}, \text{ and } \frac{b}{c}=\frac{\sin. B}{\sin. C}.$$

Therefore $$\frac{a+b}{c}=\frac{\sin. A+\sin. B}{\sin. C}=\frac{\sin. A+\sin. B}{\sin. (A+B)}.$$

By Trigonometry, Art. 75,

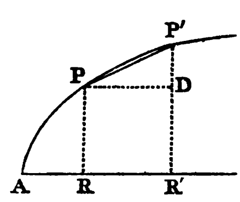
$$\frac{\sin. A+\sin. B}{\sin. (A+B)}=\frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)}.$$

Hence $$\frac{a+b}{c}=\frac{\cos. \frac{1}{2}(A-B)}{\cos. \frac{1}{2}(A+B)}.$$

Conceive now the points A and B to approach each other, and the arc ADB to decrease continually, the angles A and B will manifestly both decrease, and they may become less than any assignable angle whatever; therefore $A-B$ and $A+B$ both approach continually to 0; and $\cos. \frac{1}{2}(A-B)$ and $\cos. \frac{1}{2}(A+B)$ approach to unity, which is their common limit. Hence the limit of the ratio of $a+b$ to c is a ratio of equality; and as the arc ADB can not be greater than $a+b$, nor less than c, much more is the limit of the ratio of the arc to the chord a ratio of equality.

PROPOSITION VIII.—THEOREM.

(251.) *The differential of the arc of a curve referred to rectangular co-ordinates, is equal to the square root of the sum of the squares of the differentials of the co-ordinates.*



We have found, Art. 250, that the limit of the ratio of the chord and arc of a curve is unity; hence the differential of an arc is equal to the differential of its chord.

Let x represent any abscissa of a curve, AR for example, and y the corresponding ordinate PR. If now we give to x any arbitrary increment h, and make $RR'=h$, the value of y will become equal to $P'R'$, which we will represent

by y'. If we draw PD parallel to the axis AR′, we shall have the chord $PP'=\sqrt{PD^2+P'D^2}=\sqrt{h^2+P'D^2}$.

But $P'D=y'-y=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+$ other terms involving higher powers of h.

Substituting this value of P′D in the expression for the chord, we have

$$PP'=\sqrt{h^2+\frac{dy^2h^2}{dx^2}+\text{, etc.}},$$

$$=h\sqrt{1+\frac{dy^2}{dx^2}+\text{, etc.}}$$

Therefore $$\frac{PP'}{h}=\sqrt{1+\frac{dy^2}{dx^2}+\text{, etc.}},$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174.

In this case the chord becomes equal to the arc, which we will represent by z, and the terms omitted in the second member of the equation containing h disappear; hence

$$\frac{dz}{dx}=\sqrt{1+\frac{dy^2}{dx^2}},$$

and, multiplying by dx, $dz=\sqrt{dx^2+dy^2}$.

(252.) To determine the differential of the arc of a circle, take the equation

$$x^2+y^2=R^2;$$

whence $$xdx+ydy=0, \text{ or } dy=-\frac{xdx}{y},$$

and $$dz=\sqrt{dx^2+\frac{x^2dx^2}{y^2}}=\frac{dx}{y}\sqrt{x^2+y^2}.$$

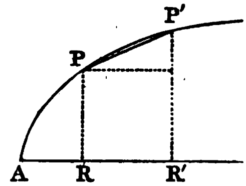
But $$\sqrt{x^2+y^2}=R, \text{ and } y=\sqrt{R^2-x^2}.$$

Hence $$dz=\frac{Rdx}{\sqrt{R^2-x^2}}.$$

PROPOSITION IX.—THEOREM.

(253.) *The differential of the area of a segment of any curve referred to rectangular co-ordinates, is equal to the ordinate into the differential of the abscissa.*

Let APR be a surface bounded by the straight lines AR,



PR, and the arc AP of a curve; it is required to find the differential of its area.

Let x represent the abscissa AR, and y the corresponding ordinate PR. If we give to x an increment h, and make $RR'=h$, the value of y will become $P'R'$, which we will represent by y'.

Since the limit of the ratio of the chord and arc of a curve is unity, the limit of the ratio of the area included by the ordinates PR, P′R′ and the arc PP′, to the trapezoid included by the same ordinates and the chord PP′, must be a ratio of equality.

Now the trapezoid $PRR'P'=RR'\times\frac{1}{2}(PR+P'R')=\frac{1}{2}h(y+y')$.

Hence $$\frac{PRR'P'}{h}=\frac{1}{2}(y+y').$$

But $$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+, \text{ etc.}$$

Hence $$\tfrac{1}{2}(y'+y)=y+\frac{dy}{dx}\frac{h}{2}+, \text{ etc.};$$

that is, $$\frac{PRR'P'}{h}=y+\frac{dy}{dx}\frac{h}{2}+, \text{ etc.},$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174.

But in this case, all the terms in the second member of the equation which contain h disappear, and representing the area of the segment by s, we have

$$\frac{ds}{dx}=y, \text{ or } ds=ydx.$$

(254.) Ex. To find the differential of the area of a circular segment, take the equation

$$y^2=R^2-x^2.$$

Whence $$y=\sqrt{R^2-x^2}.$$

Hence $$ds=ydx=dx\sqrt{R^2-x^2}.$$

The equation of the circle, when the origin of co-ordinates is placed on the circumference, is

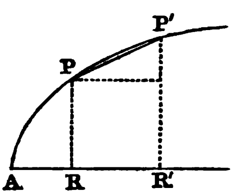
$$y=\sqrt{2rx-x^2},$$

and hence the differential of the area becomes

$$dx\sqrt{2rx-x^2}.$$

PROPOSITION X.—THEOREM.

(255.) *The differential of a surface of revolution, is equal to the circumference of a circle perpendicular to the axis, multiplied by the differential of the arc of the generating curve.*



Let the curve APP' be revolved about the axis of X, it will generate a surface of revolution; and it is required to find the differential of this surface.

Put $AR=x$ and $PR=y$. If we give to x an increment $h=RR'$, the value of y will become P'R', which we will represent by y'.

In the revolution of the curve APP', the points P and P' will describe the circumferences of two circles, and the chord PP' will describe the convex surface of a frustum of a cone. Also since the limit of the ratio of the chord and arc of a curve is unity, the limit of the ratio of the surface described by the chord to the surface described by the arc must be a ratio of equality.

Now the surface described by the chord PP' is equal to

$$\frac{PP'}{2}\times(circ.\ PR+circ.\ P'R'), \text{ Geom., Prop. IV., B. X.,}$$

which equals $$\frac{PP'}{2}(2\pi y+2\pi y'),$$

or $$PP'\times\pi(y+y').$$

Hence $$\frac{\textit{the surface of frustum}}{PP'}=\pi(y+y').$$

But $$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+, \text{ etc.},$$

and $$y'+y=2y+\frac{dy}{dx}h+, \text{ etc.}$$

Hence $$\frac{\textit{the surface of frustum}}{PP'}=\pi(2y+\frac{dy}{dx}h+, \text{ etc.})$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174.

But in this case, all the terms in the second member of the equation which contain h disappear, and representing the arc

AP by z, and the surface described by the arc AP by S, we have

$$\frac{dS}{dz}=2\pi y, \text{ or } dS=2\pi y dz;$$

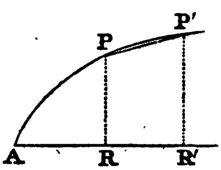
and, by substituting for dz its value, Art. 251, we have

$$dS=2\pi y(dx^2+dy^2),$$

where $2\pi y$ is the circumference of the circle described by the point P.

PROPOSITION XI.—THEOREM.

(256.) *The differential of a solid of revolution is equal to the area of a circle perpendicular to the axis, multiplied by the differential of the abscissa of the generating curve.*



Let the surface APR be revolved about the axis of X, it will generate a solid of revolution, and it is required to find its differential.

Put AR$=x$ and PR$=y$. If we give to x an increment $h=$RR$'$, the value of y will become P$'$R$'$, which we will represent by y'.

In the revolution of the surface AP$'$R$'$, the trapezoid PRR$'$P$'$ will describe the frustum of a cone, and the limit of its ratio to the solid described by the surface included by the ordinates PR, P$'$R$'$, and the arc PP$'$, is a ratio of equality.

Now the solidity of the frustum described by the trapezoid PRR$'$P$'$, Geom., Prop. VI., B. X., is

$$\tfrac{1}{3}\pi\times RR'(PR^2+P'R'^2+PR\times P'R'),$$

or

$$\tfrac{1}{3}\pi h(y^2+y'^2+yy').$$

Hence $$\frac{\textit{the solidity of the frustum}}{h}=\tfrac{1}{3}\pi(y^2+y'^2+yy').$$

But $$y'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+, \text{ etc.}$$

Hence $$y'^2=y^2+\frac{dy}{dx}hy+, \text{ etc.}$$

Also, $$yy'=y^2+\frac{dy}{dx}hy+, \text{ etc.}$$

Therefore $$y^2+y'^2+yy'=3y^2+\frac{2dy}{dx}hy+, \text{ etc.};$$

that is, $$\frac{\textit{solidity of frustum}}{h}=\tfrac{1}{3}\pi(3y^2+\frac{2dy}{dx}hy+\text{, etc.}),$$

which expresses the ratio of the increment of the function to that of the variable, and we must find the limit of this ratio by making the increment equal to zero, Art. 174.

But in this case, all the terms in the second member of the equation which contain h disappear, and representing the volume of the solid generated by V, we have

$$\frac{dV}{dx}=\pi y^2,$$

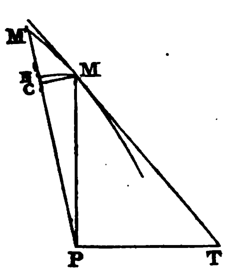
or $$dV=\pi y^2dx,$$

where πy^2 is the area of the circle described by PR.

DIFFERENTIAL OF THE ARC AND AREA OF A POLAR CURVE.

PROPOSITION XII.—THEOREM.

(257.) *The differential of an arc of a polar curve, is equal to the square root of the sum of the squares of the differential of the radius vector, and of the product of the radius vector by the differential of the measuring arc.*



Let PM, PM′ be two radius vectors of a polar curve, and let MC be drawn from M perpendicular to PM′. Then, in the right-angled triangle M′CM, we have

$$\textit{chord}\ \text{M'M}=\sqrt{\text{M'C}^2+\text{CM}^2}.$$

Also, $$\frac{\text{CM}}{\text{CM'}}=\text{tang. CM'M}.$$

Therefore

$$\frac{\textit{chord}\ \text{M'M}}{\text{M'C}}=\sqrt{1+\text{tang.}^2\ \text{CM'M}}.$$

We must now find the limit of this ratio, by making the increment of the radius vector equal to zero. The limit of the ratio of the chord MM′ to the arc MM′ is unity, Art. 250. Also, M′C approaches to M′N, which is the increment of the radius vector, and the limit of their ratio is unity; and the angle CM′M becomes PMT, which is equal to $\frac{rdt}{dr}$, Prop. V., Cor. Hence, representing the arc by z, we have

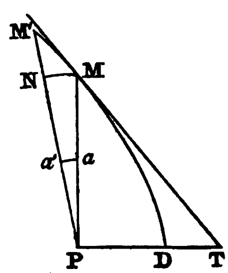
$$\frac{dz}{dr}=\sqrt{1+\frac{r^2dt^2}{dr^2}},$$

or $$dz=\sqrt{dr^2+r^2dt^2},$$

which is the differential of the arc of a polar curve.

PROPOSITION XIII.—THEOREM.

(258.) *The differential of the area of a segment of a polar curve, is equal to the differential of the measuring arc, multiplied by half the square of the radius vector.*



Let PMD be any segment of a polar curve, and let the measuring arc receive a small increment aa'; the increment of the area will be PMM′.

The area of the sector PMN, Geom., Prop. XII., Cor. B. VI., is equal to $\text{MN}\times\frac{\text{PM}}{2}$.

And since $aa' : \text{MN} :: 1 : \text{PM}$,

$$aa'=\frac{\text{MN}}{\text{PM}}.$$

Therefore $$\frac{\textit{sector}\ \text{PMN}}{aa'}=\frac{\text{PM}^2}{2}.$$

Now since the limit of the ratio of PM′ to PM is a ratio of equality, the limit of the ratio of PMM′ to the circular sector PMN is a ratio of equality. Taking the value of this ratio when the increment is equal to zero, and representing the segment by s, we have

$$\frac{ds}{dt}=\frac{r^2}{2},$$

or $$ds=\frac{r^2dt}{2},$$

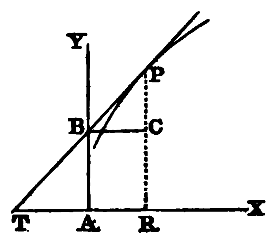
which is the differential of the area of a segment of a polar curve.

ASYMPTOTES OF CURVES.

(259.) An *asymptote* of a curve is a line which continually approaches the curve, and becomes tangent to it at an infinite distance from the origin of co-ordinates.

In some curves the distance between the origin of the co-

ordinates and the point in which the tangent meets the axes increases continually with the abscissa, so that when the abscissa x becomes infinite, this distance is infinite. In other curves, even when the abscissa becomes infinite, the tangent cuts the axes at a finite distance from the origin. It is then called an asymptote to the curve.



Let A be the origin of co-ordinates, and let TP be a tangent to the curve at a point whose co-ordinates are AR$=x$ and PR$=y$. If from the subtangent TR, which equals $y\frac{dx}{dy}$, the abscissa AR be subtracted, the remainder,

$$\mathrm{AT}=y\frac{dx}{dy}-x, \qquad (1)$$

is the general expression for AT, the distance from the origin at which the tangent intersects the axis of X.

Also, $$\mathrm{PC}=\mathrm{BC}\ \mathrm{tang.}\ \mathrm{PBC}=\frac{dy}{dx}x.$$

Hence $$\mathrm{AB}=\mathrm{PR}-\mathrm{PC}=y-\frac{dy}{dx}x, \qquad (2)$$

which is a general expression for AB, the distance from the origin at which the tangent intersects the axis of Y.

(260.) If, when x and y become infinite, either of the expressions (1) and (2) reduces to a finite quantity, we may conclude that the curve has asymptotes; but if both be infinite, then the curve has no asymptotes. If both the expressions are finite, the asymptote will be inclined to both the co-ordinate axes; if one of the values becomes finite and the other infinite, the asymptote will be parallel to one of the co-ordinate axes; if both become zero, the asymptote will pass through the origin of co-ordinates.

(261.) Ex. 1. It is required to determine whether the hyperbola has asymptotes.

The equation of the hyperbola, when the origin of co-ordinates is at the center, is (Art. 98)

$$y^2=\frac{\mathrm{B}^2}{\mathrm{A}^2}(x^2-\mathrm{A}^2).$$

Differentiating, we find

$$\frac{dx}{dy}y=\frac{A^2y^2}{B^2x}=\frac{x^2-A^2}{x}.$$

Therefore $$AT=v\frac{dx}{dy}-x=-\frac{A^2}{x}.$$

The expression $-\frac{A^2}{x}$ represents the distance from the origin of co-ordinates at which the tangent intersects the axis of X.

When x is supposed infinite, this expression becomes equal to zero. Hence the hyperbola has asymptotes which pass through the center.

Ex. 2. It is required to determine whether the parabola has asymptotes.

The equation of the parabola is

$$y^2=2px.$$

Differentiating, we find

$$y\frac{dx}{dy}=\frac{y^2}{p}=2x.$$

Hence $$AT=y\frac{dx}{dy}-x=x.$$

When x is infinite, this quantity becomes infinite; therefore the parabola has no asymptotes.

Ex. 3. The equation of the logarithmic curve is

$$x=\log.\ y.$$

or $$y=a^x.$$

If x be taken infinite and negative, then

$$y=\frac{1}{a^\infty}=0;$$

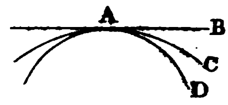
that is, *the axis of abscissas is an asymptote to the curve.* See fig., page 108.

SECTION VI.

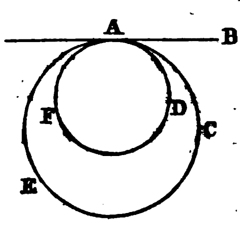
RADIUS OF CURVATURE—EVOLUTES OF CURVES.

(262.) The *curvature* of a curve is its deviation from the tangent; and of two curves that which departs most rapidly from its tangent, is said to have the greatest curvature.

Thus, of the two curves AC, AD, having the common tangent AB, the latter departs most rapidly from the tangent, and is said to have the greatest curvature.



(263.) The curvature of the circumference of a circle is evidently the same at all of its points, and also in all circumferences described with equal radii, since the deviation from the tangent is the same; but of two different circumferences, that one curves the most which has the least radius. Thus, the circumference ADF departs more rapidly from the tangent line AB than the circumference ACE, and this deviation increases as the radius decreases, and the reverse. In different circumferences the curvature is measured by the angle formed by two radii drawn through the extremities of an arc of given length.



Proposition I.—Theorem.

(264.) *The curvature in two different circles varies inversely as their radii.*

Let R and R′ represent the radii of two circles, A the length of a given arc measured on the circumference of each; α the angle formed by the two radii drawn through the extremities of the arc in the first circle, and α' the angle formed by the corresponding radii of the second. Then, by Geom., Prop. XIV., B. III., we have

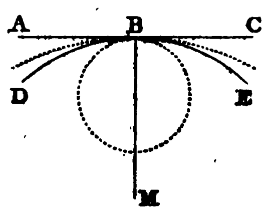
$$2\pi R : A :: 360° : \alpha; \text{ whence } \alpha = \frac{360A}{2\pi R};$$

and $\qquad 2\pi R' : A :: 360° : \alpha'$; whence $\alpha' = \dfrac{360A}{2\pi R'}$.

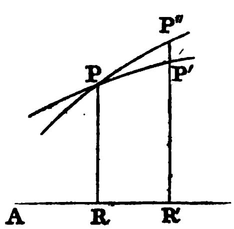
Therefore $$\alpha : \alpha' :: \frac{360A}{2\pi R} : \frac{360A}{2\pi R'},$$

or $$\alpha : \alpha' :: \frac{1}{R} : \frac{1}{R'};$$

that is, *the curvature in two different circles varies inversely as their radii.*



(265.) Let DBE be any curve line, ABC a tangent at the point B, and BM a normal at the same point, then will ABC be a tangent to the circumference of every circle passing through B, and having its center in the line BM. The curve DBE may, therefore, be touched by an infinite number of circles at the same point B. Some of these circles, having a greater curvature than the curve, fall wholly within it; while others, having a less degree of curvature, fall between the curve and the tangent. Of this infinite number of circles, there is one which coincides most intimately with the curve, and is hence called the *osculatory circle*, or *circle of curvature*, and its radius is called the *radius of curvature* of the curve. The osculatory circle may be found in the following manner.



(266.) Let there be two curves which meet at the point P, and let us designate the co-ordinates of one curve by x and y, and the co-ordinates of the second curve by x' and y'. If we suppose x to receive an increment and become $x+h$, we shall have

$$P'R' = y + \frac{dy}{dx}h + \frac{d^2y}{dx^2}\frac{h^2}{2} + \frac{d^3y}{dx^3}\frac{h^3}{2.3} +, \text{etc.,} \qquad (1)$$

$$P''R' = y' + \frac{dy'}{dx'}h + \frac{d^2y'}{dx'^2}\frac{h^2}{2} + \frac{d^3y'}{dx'^3}\frac{h^3}{2.3} +, \text{etc.} \qquad (2)$$

But since the point P is common to the two curves, we must have $$y = y'.$$

Also, since the first differential coefficient represents the tangent of the angle which a tangent line makes with the axis of

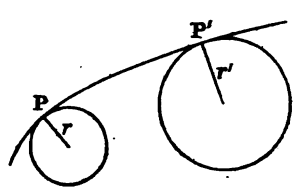
abscissas, if we suppose the two curves to have a common tangent at P, we must have

$$\frac{dy}{dx}h=\frac{dy'}{dx'}h.$$

Now if all the terms in the first of these developments are equal to the corresponding terms in the other, the curves will be identical; and the greater the number of terms which are equal in the two developments, the more intimate will be the contact of the curves.

Since the equation of a circle is of the second degree, it can have but two differential coefficients; and therefore *a circle will coincide most nearly with a given curve, when its first and second differential coefficients are equal to the first and second differential coefficients of the equation of the curve.*

(267.) Since the contact of the osculatory circle with a curve is so intimate, its curvature is regarded as measured by means of the osculatory circle. Thus, if we assume two points in the curve PP′, and find the radii r and r' of the circles which are osculatory at these points, we shall have



curvature at P : *curvature* P′ $:: \frac{1}{r} : \frac{1}{r'}$;

that is, *the curvature at different points varies inversely as the radius of the osculatory circle.*

PROPOSITION II.—THEOREM.

(268.) *The radius of curvature at any point of a given curve is equal to*

$$\frac{dz^3}{dxd^2y},$$

where x and y are the co-ordinates of the given point, and z the arc of the given curve.

The general equation to the circle, Art. 38, is

$$(x-a)^2+(y-b)^2=R^2,$$

where a and b are the co-ordinates of the center of the circle, and R is the radius.

Differentiating this equation, and dividing by 2, we have

$$(x-a)dx+(y-b)dy=0.$$

Differentiating again, regarding dx as constant, we obtain

$$dx^2+dy^2+(y-b)d^2y=0.$$

Whence $$y-b=-\frac{dx^2+dy^2}{d^2y}, \quad (1)$$

and $$x-a=\frac{dy}{dx}\left(\frac{dx^2+dy^2}{d^2y}\right). \quad (2)$$

Substituting these values in the equation of the circle, we have

$$R^2=\frac{dy^2}{dx^2}\left(\frac{dx^2+dy^2}{d^2y}\right)^2+\left(\frac{dx^2+dy^2}{d^2y}\right)^2,$$

or $$R^2=\frac{(dx^2+dy^2)^3}{(dxd^2y)^2};$$

whence $$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}.$$

If z denote the arc of the given curve, then, Art. 251,

$$dz^2=dx^2+dy^2;$$

and the above expression for R becomes

$$R=\frac{dz^3}{dxd^2y}, \quad (3)$$

which is a general expression for the value of the radius of the osculatory circle.

(269.) To find the radius of curvature for any particular curve, we must differentiate the equation of the curve twice, and substitute the values of dx, dy, and d^2y in the preceding expression for R. If the radius of curvature for a particular point of the curve is required, we must substitute for x and y the coordinates of the given point.

PROPOSITION III.—THEOREM.

(270.) *The radius of curvature at any point of a conic section, is equal to the cube of the normal divided by the square of half the parameter.*

The general equation of the conic sections, Art. 132, is

$$y^2=mx+nx^2;$$

whence $$dy=\frac{(m+2nx)dx}{2y},$$

and $$dx^2+dy^2=\frac{[4y^2+(m+2nx)^2]dx^2}{4y^2}.$$

Also,
$$d^2y=\frac{2nydx^2-(m+2nx)dxdy}{2y^2},$$
$$=\frac{[4ny^2-(m+2nx)^2]dx^2}{4y^3}.$$

Substituting these values in the equation
$$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y},$$
we obtain
$$R=\frac{[4(mx+nx^2)+(m+2nx)^2]^{\frac{3}{2}}}{2m^2},$$
and dividing both terms of the fraction by 8, it becomes
$$R=\frac{(\sqrt{mx+nx^2+\frac{1}{4}(m+2nx)^2})^3}{\frac{1}{4}m^2}.$$

The numerator of this expression is the cube of the normal, Art. 237, and the denominator is the square of half the parameter, Art. 132; that is, the radius of curvature is equal to the cube of the normal divided by the square of half the parameter.

(271.) *Cor.* 1. The radii of curvature at different points of the same conic section, are to each other as the cubes of the corresponding normals.

Cor. 2. If we make $x=0$, we have
$$R=\frac{m}{2}=\text{one half the parameter};$$
that is, *the radius of curvature at the vertex of the major axis of any conic section, is equal to half the parameter of that axis.*

Cor. 3. If it be required to find the radius of curvature at the vertex of the minor axis of an ellipse, we make
$$m=\frac{2B^2}{A},\ n=-\frac{B^2}{A^2},\ \text{and}\ x=A,$$
which gives, after reducing,
$$R=\frac{A^2}{B};$$
that is, *the radius of curvature at the vertex of the minor axis of an ellipse, is equal to one half the parameter of that axis.*

Cor. 4. In the case of the parabola in which $n=0$, the general value of the radius of curvature becomes

$$R = \frac{(m^2+4mx)^{\frac{3}{2}}}{2m^2},$$

when $x=0$, $R=\frac{m}{2}$, which is the radius of curvature at the vertex of the parabola.

Ex. 1. Required the length of the radius of curvature for a point in a parabola whose abscissa is 9, and ordinate 6.

Ans.

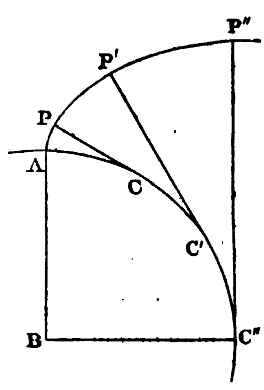
Ex. 2. Required the radius of curvature at the vertex of the major axis of an ellipse whose major axis is 10 inches, and minor axis 6 inches. *Ans.*

Ex. 3. Required the radius of curvature at the vertex of the minor axis of the same ellipse.

Ans.

EVOLUTES OF CURVES.

(272.) An *evolute* is a curve from which a thread is supposed to be unwound or *evolved*, its extremity at the same time describing another curve called the *involute*.



Thus, let ACC′C″ be any curve, and suppose a thread, fastened to it at some point beyond C″, is drawn tight to the curve. Let it now be gradually unwound from the curve, keeping it always tight. While the portion between A and C″ is unwinding, its extremity will describe upon the plane some curve line APP′P″, the nature of which will depend on the properties of the other curve.

The curve ACC′C″ about which the thread is wrapped, is called the *evolute* of the curve APP″ generated by the extremity of the thread; and the latter curve is called the *involute* of the former.

(273.) From the manner in which a curve is generated from its evolute, we may derive the following conclusions:

1st. The portion of the thread PC, which is disengaged from the evolute, is a *tangent* to it at C.

2d. *A tangent to the evolute curve at* C *is perpendicular to*

the involute at the point P; and any point C of the evolute may be considered as a momentary center, and the line CP as the radius of a circle which the point P is describing when the point of contact of the tangent and curve is at C. The points C, C′, C″ are therefore the centers of curvature of the points P, P′, P″; and PC, P′C′, P″C″ are the radii of curvature of the involute at the points C, C′, and C″.

3d. *The radius of curvature* PC *is equal to the arc* AC *of the evolute,* reckoned from the point A, where the curve commences.

(274.) Hence, if we suppose an osculatory circle to be drawn at each of the points of the curve A, P, P′, P″, the centers of all these circles will be found upon the curve ACC′C″. The equation of the evolute is therefore the equation which expresses the relation between the centers of all the osculatory circles of the involute.

The general equation of the circle, Art. 38, is

$$(x-a)^2+(y-b)^2=R^2, \qquad (1)$$

where a and b denote the co-ordinates of the center of the circle.

To determine the equation of the evolute, we must find the relation of a to b in equation (1), regarding a and b as the co-ordinates of the center of the circle of curvature, and consequently the co-ordinates of the evolute.

But we have found, Art. 268, for the circle of curvature

$$y-b=-\frac{dx^2+dy^2}{d^2y}, \qquad (2)$$

and

$$x-a=-\frac{dy}{dx}(y-b), \qquad (3)$$

and combining these with the equation of the involute curve, we may obtain an equation from which x and y are eliminated.

We must therefore differentiate the equation of the involute twice; deduce the values of dy and d^2y, and substitute them in equations (2) and (3); two new equations will thus be obtained involving a, b, x, and y.

Combine these equations with the equation of the involute, and eliminate x and y; the resulting equation will contain only a and b, and constants, and will be the equation of the evolute curve.

Ex. Let it be required to find the equation of the evolute of the common parabola.

The equation of the involute is

$$y^2=2px;$$

whence $$\frac{dy}{dx}=\frac{p}{y}.$$

Also, $$dy^2=\frac{p^2dx^2}{y^2}, \text{ and } d^2y=-\frac{p^2dx^2}{y^3}.$$

Substituting these values in equations (2) and (3), and reducing, we have

$$y-b=\frac{y^3}{p^2}+y; \text{ whence } -b=\frac{y^3}{p^2},$$

and $$x-a=-\frac{y^2}{p}-p.$$

Substituting for y its value $(2px)^{\frac{1}{2}}$, we have

$$-b=\frac{2^{\frac{3}{2}}x^{\frac{3}{2}}}{p^{\frac{1}{2}}}; \text{ and } x-a=-2x-p.$$

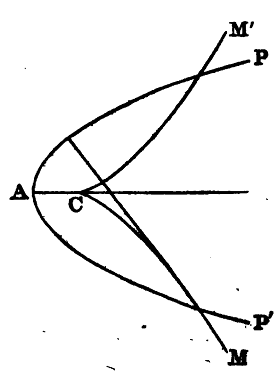
From this last equation we derive

$$x=\frac{a-p}{3}.$$

Substituting this value of x in the preceding equation, we have

$$b^2=\frac{8}{27}\,\frac{(a-p)^3}{p},$$

which is the equation of the evolute, and shows it to be *the semi-cubical parabola*, Art. 136.



If we make $b=0$, we have

$$a=p,$$

and hence the evolute meets the axis of abscissas, at a distance AC from the origin equal to half the parameter.

If we transfer the origin of co-ordinates from A to C, the above equation reduces to

$$b^2=\frac{8}{27p}a^3.$$

Since every value of a gives two equal values of b with contrary signs, the curve is symmetrical

with respect to the axis of abscissas. The evolute CM corresponds to the part AP of the involute, and CM′ to the part AP′.

PROPERTIES OF THE CYCLOID.

(275.) We have found, Art. 140, the equation of the cycloid to be

$$x = \text{arc whose versed sine is } y - \sqrt{2ry - y^2}.$$

The properties of the cycloid are most easily deduced from its differential equation, which is obtained by differentiating both members of the transcendental equation.

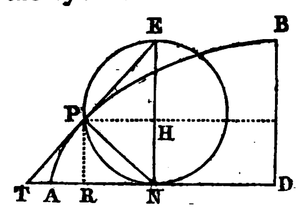
By Art. 226, $d(\text{arc whose versed sine is } y) = \frac{rdy}{\sqrt{2ry-y^2}}$.

By Art. 187, $d(-\sqrt{2ry-y^2}) = -\frac{rdy-ydy}{\sqrt{2ry-y^2}}$.

Hence $$dx = \frac{rdy}{\sqrt{2ry-y^2}} - \frac{rdy-ydy}{\sqrt{2ry-y^2}},$$

or $$dx = \frac{ydy}{\sqrt{2ry-y^2}},$$

which is the differential equation of the cycloid.



(276.) If we substitute the preceding value of dx in the formulas of Articles 233–6, we shall obtain the values of the tangent and subtangent, normal and subnormal of the cycloid. They are

$$\text{subtangent TR} = y\frac{dx}{dy} = \frac{y^2}{\sqrt{2ry-y^2}},$$

$$\text{tangent PT} = y\sqrt{1+\frac{dx^2}{dy^2}} = \frac{y\sqrt{2ry}}{\sqrt{2ry-y^2}},$$

$$\text{subnormal RN} = y\frac{dy}{dx} = \sqrt{2ry-y^2},$$

$$\text{normal PN} = y\sqrt{1+\frac{dy^2}{dx^2}} = \sqrt{2ry}.$$

The subnormal RN is equal to PH of the generating circle, since each is equal to $\sqrt{2ry-y^2}$; hence the normal PN and the diameter EN intersect the base of the cycloid at the same point.

PROPOSITION IV.—THEOREM.

(277.) *The radius of curvature corresponding to any point of the cycloid, is equal to double the normal.*

If we differentiate again the differential equation of the cycloid,

$$dx=\frac{ydy}{\sqrt{2ry-y^2}},$$

regarding dx as constant, we obtain

$$0=(yd^2y+dy^2)\sqrt{2ry-y^2}-\frac{ydy(rdy-ydy)}{\sqrt{2ry-y^2}}.$$

Clearing of fractions, uniting terms, and dividing by y, we have

$$0=(2ry-y^2)d^2y+rdy^2;$$

whence

$$d^2y=-\frac{rdy^2}{2ry-y^2}=-\frac{rdx^2}{y^2}.$$

Substituting the values of dy and d^2y in the expression for the radius of curvature, Art. 268,

$$R=\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxd^2y}$$

we obtain

$$R=\frac{\left(\frac{2rdx^2}{y}\right)^{\frac{3}{2}}}{\frac{rdx^2}{y^2}}=2^{\frac{3}{2}}r^{\frac{1}{2}}y^{\frac{1}{2}}=2\sqrt{2ry}.$$

But we have found the normal, Art. 276, equal to $\sqrt{2ry}$; hence the radius of curvature is equal to double the normal at the point of contact.

PROPOSITION V.—THEOREM.

(278.) *The evolute of a cycloid is an equal cycloid.*

To obtain the equation of the evolute, we must substitute the values of dy and d^2y, already found for the cycloid, in equations (2) and (3) of Art. 274. We thus obtain

$$y-b=-\frac{dx^2+dy^2}{d^2y}=\frac{dx^2+\frac{(2ry-y^2)dx^2}{y^2}}{\frac{rdx^2}{y^2}},$$

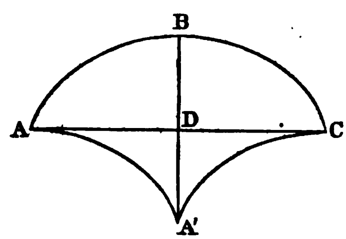
$$=\frac{y^2+2ry-y^2}{r}=2y.$$

Whence $$y=-b.$$

Also, $$x-a=-\frac{dy}{dx}(y-b)=-\frac{\sqrt{2ry-y^2}}{y}\times 2y,$$
$$=-2\sqrt{2ry-y^2}.$$

Whence $$x=a-2\sqrt{2ry-y^2}.$$

Substituting these values of x and y in the transcendental equation of the cycloid, Art. 140,
$$x=arc(versed\ sine=y)-\sqrt{2ry-y^2},$$
we obtain
$$a-2\sqrt{-2rb-b^2}=arc(versed\ sine=-b)-\sqrt{-2rb-b^2},$$
or $$a=arc(versed\ sine=-b)+\sqrt{-2rb-b^2};$$
which is the transcendental equation of the evolute referred to the primitive origin and the primitive axes.



This is also the equation of a cycloid whose generating circle is equal to that of the given one, and whose vertex coincides with the extremity of the base, lying, however, below the base, as appears by substituting $-b$ for y in the equation of Art. 140.

Thus, the evolute AA′ of the cycloid is an equal cycloid; the arc AA′ is identical with AB, and the vertex B is transferred to A.

SECTION VII.

ANALYSIS OF CURVE LINES.

(279.) If it was possible to resolve an equation of any degree, we might follow the course of a curve represented by any Algebraic equation, by methods explained in Analytical Geometry. By assigning to the independent variable different values, both positive and negative, we could determine any number of points of the curve at pleasure.

The Differential Calculus enables us to abridge this investigation, and may be employed even when the equation of the curve is of so high a degree that we are unable to obtain a general expression for one of the variables in terms of the other.

The first object aimed at in such an analysis, is to discover those points of a curve which present some peculiarity; such as the point at which the tangent is parallel or perpendicular to the axis of abscissas. Such points have been named singular points. A *singular point* of a curve is one which is distinguished by some remarkable property not enjoyed by the other points of the curve immediately adjacent.

Proposition I.—Theorem.

(280.) *For a point at which the tangent to a curve is parallel to the axis of abscissas, the first differential coefficient is equal to zero.*

For the first differential coefficient expresses the value of the tangent of the angle which the tangent line forms with the axis of abscissas, Art. 201; and when this line is parallel with the axis, the angle which it forms with the axis is zero, and its tangent is zero.

Proposition II.—Theorem.

(281.) *For a point at which the tangent to a curve is perpendicular to the axis of abscissas, the first differential coefficient is equal to infinity.*

For the first differential coefficient expresses the value of

the tangent of the angle which the tangent line forms with the axis of abscissas; and when this angle is 90 degrees, its tangent is infinite.

Ex. 1. It is required to determine at what point the tangent to the circumference of a circle is parallel to the axis, and where it is perpendicular.

Take the equation

$$x^2+y^2=R^2.$$

By differentiating, we obtain

$$\frac{dy}{dx}=-\frac{x}{y},$$

and placing this equal to zero, we find

$$x=0.$$

But when $x=0$, we have

$$y=\pm R;$$

hence the tangent is parallel to the axis of abscissas at the two points where the circumference intersects the axis of ordinates.

If we make

$$\frac{dy}{dx}=-\frac{x}{y}=\infty, \text{ or } -\frac{y}{x}=0,$$

we find $y=0$. But when $y=0$, we have

$$x=\pm R;$$

that is, the tangent is perpendicular to the axis of abscissas at the two points where the circumference intersects the axis of abscissas.

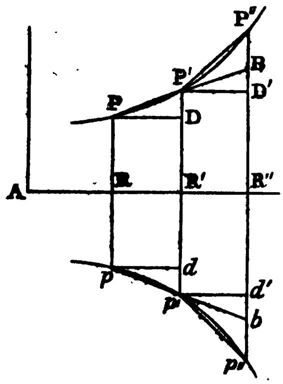
Ex. 2. It is required to determine at what point the tangent to a cycloid is parallel to the base, and when it is perpendicular to the base.

PROPOSITION III.—THEOREM.

(282.) *If a curve is convex toward the axis of abscissas, the ordinate and second differential coefficient will have the same sign.*

Let PP′P″ be a curve convex toward the axis of abscissas; and let x and y be the co-ordinates of the point P. Let x be increased by any arbitrary increment RR′, which we will rep-

resent by h, and take $R'R''$ also equal to h. Draw the ordinates $P'R'$, $P''R''$; draw the line PP', and produce it to B; join P' and P'', and draw PD, $P'D'$ parallel to AR. We shall then have



$$PR=y,$$

$$P'R'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+, \text{ etc.} \quad (1)$$

Also,

$$P''R''=y+\frac{dy}{dx}2h+\frac{d^2y}{dx^2}\frac{4h^2}{2}+, \text{ etc.} \quad (2)$$

Hence $$P'D=P'R'-PR=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+, \text{ etc.} \quad (3)$$

Subtracting equation (1) from equation (2), we have

$$P''D'=P''R''-P'R'=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{3h^2}{2}+, \text{ etc.} \quad (4)$$

Subtracting equation (3) from equation (4), remembering that BD' is equal to $P'D$, we have

$$P''B=P''D'-P'D=\frac{d^2y}{dx^2}h^2+, \text{ etc.} \quad (5)$$

Now when we suppose h to be taken indefinitely small, the sign of the second member of equation (5) will depend upon that of the first term, and since the first member of the equation is positive, the second must also be positive; that is, the second differential coefficient is positive; and the ordinate, being situated above the axis of abscissas, is also positive.

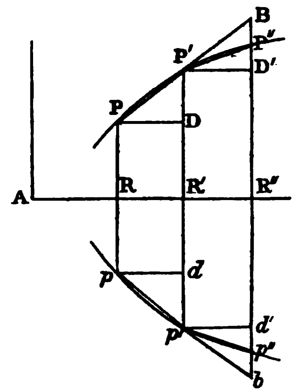
If the curve is below the axis of abscissas, we shall have

$$-p''b=p''d'-p'd=\frac{d^2y}{dx^2}h^2+, \text{ etc.};$$

and since the first member of this equation is negative, the second will also be negative; that is, the second differential coefficient is negative. Whence we conclude that if the curve is convex toward the axis of abscissas, the second differential coefficient will be positive when the ordinate is positive, and negative when the ordinate is negative.

PROPOSITION IV.—THEOREM.

(283.) *If a curve is concave toward the axis of abscissas, the ordinate and second differential coefficient will have contrary signs.*



Let PP′P″ be a curve concave toward the axis of abscissas; and let x and y be the co-ordinates of the point P. Let x be increased by any arbitrary increment RR′, which we will represent by h, and take R′R″ also equal to h. Draw the ordinates P′R′, P″R″; draw the line PP′, and produce it to B. Join P′ and P″, and draw PD, P′D′ parallel to AR. We shall then have

$$PR=y,$$

$$P'R'=y+\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\text{, etc.} \qquad (1)$$

Also,
$$P''R''=y+\frac{dy}{dx}2h+\frac{d^2y}{dx^2}\frac{4h^2}{2}+\text{, etc.} \qquad (2)$$

Hence
$$P'D=P'R'-PR=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{h^2}{2}+\text{, etc.} \qquad (3)$$

Subtracting equation (1) from equation (2), we have

$$P''D'=P''R''-P'R'=\frac{dy}{dx}h+\frac{d^2y}{dx^2}\frac{3h^2}{2}+\text{, etc.} \qquad (4)$$

Subtracting equation (3) from equation (4), remembering that BD′ is equal to P′D, we have

$$-P''B=P''D'-P'D=\frac{d^2y}{dx^2}h^2+\text{, etc.} \qquad (5)$$

Now since the first member of this equation is negative, the second member must also be negative; that is, the second differential coefficient will be negative, while the ordinate is positive.

If the curve is below the axis of abscissas, we shall have

$$+p''b=p''d'-p'd=\frac{d^2y}{dx^2}h^2+\text{, etc.,}$$

where the second differential coefficient is positive, while the ordinate is negative.

Ex. 1. It is required to determine whether the circumference of a circle is convex or concave toward the axis of abscissas.

The equation of the circle is

$$x^2+y^2=R^2;$$

whence
$$\frac{dy}{dx}=-\frac{x}{y}$$

Also,
$$\frac{d^2y}{dx^2}=-\frac{x^2+y^2}{y^3}=-\frac{R^2}{y^3},$$

which is negative when y is positive, and positive when y is negative. Hence the circumference is concave toward the axis of abscissas.

Ex. 2. It is required to determine whether the circumference of an ellipse is convex or concave toward the axis of abscissas.

(284.) DEFINITION. *A point of inflection* is a point at which a curve from being convex toward the axis of abscissas, becomes concave, or the reverse.

PROPOSITION V.—THEOREM.

For a point of inflection, the second differential coefficient must be equal to zero or infinity.

When the curve is convex toward the axis of abscissas, the ordinate and second differential coefficient have the same sign; but when the curve is concave, they have contrary signs. Hence, at a point of inflection, the second differential coefficient must change its sign. Therefore, between the positive and negative values there must be one value equal to zero or infinity; and the roots of the equation

$$\frac{d^2y}{dx^2}=0, \text{ or } \frac{d^2y}{dx^2}=\infty,$$

will give the abscissas of the points of inflection.

Having discovered that the second differential coefficient for a certain point of a curve is equal to zero or infinity, we increase and diminish successively by a small quantity h, the abscissa of this point; and if the second differential coefficient has contrary signs for these new values of x, we conclude that here is a point of inflection.

Ex. 1. Determine whether the curve whose equation is

$$y=a+(x-b)^3,$$

has a point of inflection.

By differentiating, we find

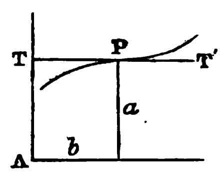
$$\frac{dy}{dx}=3(x-b)^2,$$

and

$$\frac{d^2y}{dx^2}=6(x-b).$$

When $x=b$, the first differential coefficient is zero, and the tangent is parallel to the axis of abscissas at the point whose co-ordinates are $x=b$, $y=a$.

When $x<b$, the second differential coefficient is negative; but when $x>b$, the second differential coefficient is positive; that is, the second differential coefficient changes its sign at the point of the curve of which the abscissa is $x=b$; consequently there is an inflection of the curve when $x=b$.

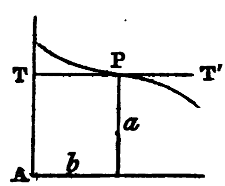


On the left of P the curve falls below the tangent line TT′, while on the right of P it runs above TT′.

Ex. 2. Determine whether the curve whose equation is

$$y=a-(x-b)^3,$$

has a point of inflection.



Ans. The curve is first convex and then concave toward the axis of abscissas, and there is an inflection at the point $x=b$.

Ex. 3. Determine whether the curve whose equation is

$$y=3x+18x^2-2x^3,$$

has a point of inflection.

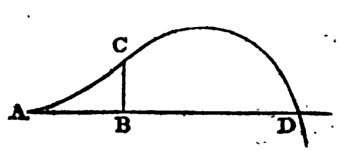
By differentiating, we find

$$\frac{dy}{dx}=3+36x-6x^2,$$

and

$$\frac{d^2y}{dx^2}=36-12x.$$

Putting the second differential coefficient equal to zero, we obtain $x=3$.



Take, therefore, AB=3, and draw the ordinate BC: C is the point of inflection. If x be between 0 and 3, $36-12x$ is posi-

tive; therefore the part AC of the curve is convex to AB; but when x is greater than 3, $36-12x$ is negative, and therefore the curve is concave toward the axis.

(285.) DEFINITION. A *multiple* point is a point at which two or more branches of a curve intersect each other.

PROPOSITION VI.—THEOREM.

For a multiple point, the first differential coefficient must have several values.

It is obvious that where two branches of a curve intersect, there must be two tangents which have different values; and since the first differential coefficient expresses the tangent of the angle which the tangent makes with the axis of abscissas, this coefficient must have as many values as there are intersecting branches.

For a multiple point, the first differential coefficient generally reduces to the form of $\frac{0}{0}$, which represents an indeterminate quantity, Algebra, Art. 130.

Ex. 1. It is required to determine whether the curve represented by the equation

$$y^2=a^2x^2-x^4,$$

has a multiple point.

Extracting the root of each member, we have

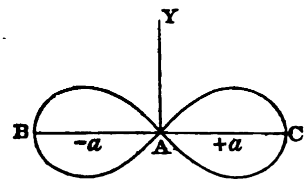
$$y=\pm x(a^2-x^2)^{\frac{1}{2}}. \qquad (1)$$

By differentiating, we obtain

$$\frac{dy}{dx}=\pm\frac{a^2-2x^2}{(a^2-x^2)^{\frac{1}{2}}}. \qquad (2)$$

We see from equation (1) that every value of x gives two values of y with contrary signs; hence the curve has two branches, which are symmetrical with respect to the axis of X. Also, when $x=\pm a$, $y=0$; that is, the curve cuts the axis of x at the points B and C, at the distances $+a$ and $-a$ from the origin. When $x=0$, $y=0$; hence the two branches intersect at the origin A, which is therefore a multiple point. At this point there are two tangents given by equation (2), which, when $x=0$, reduces to

$$\frac{dy}{dx}=\pm a.$$



Hence one tangent line makes an angle with the axis of abscissas whose tangent is $+a$, the other an angle whose tangent is $-a$.

Ex. 2. It is required to determine whether the curve represented by the equation

$$y^2=(x-a)^2(x-b),$$

has a multiple point.

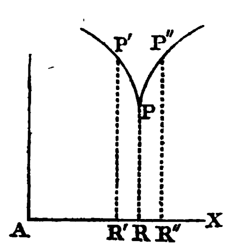
Ans. The point whose co-ordinates are $x=a$, $y=0$, is a multiple point.

(286.) DEFINITION. A *cusp* is a point at which two or more branches of a curve terminate and have a common tangent. If the branches lie on different sides of the tangent, it is called a cusp of *the first order;* if both branches lie on the same side of the tangent, it is called a cusp of *the second order.*

Since the axes of reference may be chosen at pleasure, we shall, for convenience, suppose the tangent at a cusp to be perpendicular to the axis of abscissas. If the tangent is parallel to the axis of abscissas, we have but to transpose the terms abscissa and ordinate in the two following theorems.

PROPOSITION VII.—THEOREM.

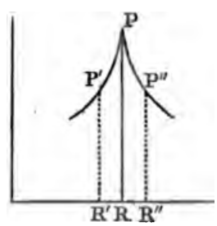
(287.) *A point of a curve at which the tangent is perpendicular to the axis of abscissas, and the contiguous ordinates on each side of that point are real, and both greater or both less than the ordinate of the given point, is a cusp of the first order*



If P be a point of curve at which the tangent is perpendicular to AX, and if the ordinates P′R′, P″R″, however near they may be taken to PR, are both *greater* than PR, it is evident that P will be the point of meeting of two branches which have PR for their common tangent, as represented in the annexed figure.

If P′R′, P″R″ are both *less* than PR, P will be the point of meeting of two branches which have PR for their common

tangent, but the branches will be situated as in the figure annexed.



Ex. 1. It is required to determine whether the curve represented by the equation

$$y=a+2(x-b)^{\frac{2}{3}},$$

has a cusp of the first order.

By differentiating, we obtain

$$\frac{dy}{dx}=\frac{4}{3(x-b)^{\frac{1}{3}}}.$$

When $x=b$, this coefficient becomes infinite, and the tangent will be perpendicular to the axis of abscissas at the point whose co-ordinates are $x=b$, $y=a$.

Let us now substitute for x, in the equation of the curve, $b+h$ and $b-h$ successively; we shall obtain in each case

$$y=a+2h^{\frac{2}{3}};$$

and hence y is *less* when $x=b$, than for the adjacent values of x either greater or less than b. Hence there is a cusp at the point whose co-ordinates are $x=b$, $y=a$.

Ex. 2. It is required to determine whether the curve represented by the equation

$$y=a-2(x-b)^{\frac{2}{3}},$$

has a cusp of the first order.

If we substitute for x, in the equation of the curve, $b+h$ and $b-h$ successively, we shall obtain in each case

$$y=a-2h^{\frac{2}{3}},$$

and hence y is *greater* when $x=b$, than for the adjacent values of x either greater or less than b. Hence there is a cusp at the point whose co-ordinates are $x=b$, $y=a$.

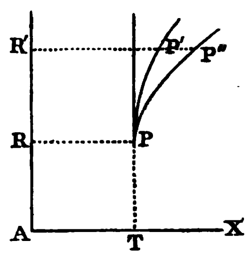
Ex. 3. It is required to determine whether the curve represented by the equation

$$x^2=y^3,$$

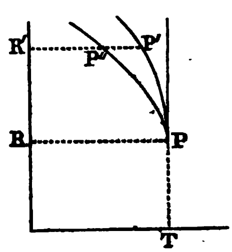
has a cusp of the first order.

PROPOSITION VIII.—THEOREM.

(288.) *A point of a curve at which the tangent is perpendicular to the axis of abscissas, and the contiguous abscissa upon one side of the given point, has two values both greater or both less than the abscissa of the given point, is a cusp of the second order.*



If P be a point of a curve at which the tangent is perpendicular to AX, and if corresponding to the ordinate AR′, there are two abscissas P′R′, P″R′, both *greater* than PR, however near they may be taken to PR, it is evident that P is the point of meeting of two branches which have PT for their common tangent, as represented in the annexed figure.



If P′R′, P″R′ are both *less* than PR, P will be the point of meeting of two branches which have PT for their common tangent, but the branches will be situated as in the figure annexed.

Ex. It is required to determine whether the curve represented by the equation

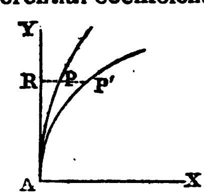
$$x=y^2\pm y^{\frac{5}{2}},$$

has a cusp of the second order.

By differentiating, we obtain

$$\frac{dy}{dx}=\frac{1}{2y\pm\frac{5}{2}y^{\frac{3}{2}}}.$$

We see from the equation of the curve that the curve has two branches, both of which pass through the origin of co-ordinates. When $y=0$, $x=0$, and the first differential coefficient reduces to infinity; and hence the axis of ordinates is tangent to both branches of the curve at the origin of co-ordinates. If y is supposed to be negative, x is imaginary; hence the curve does not extend below the axis of abscissas.



If we suppose $y=+h$, we shall have

$$x=h^2 \pm h^{\frac{5}{2}}.$$

When h is less than unity, $h^{\frac{5}{2}}$ is less than h^2, and x will have two positive values, PR and P′R; hence the point A is a cusp of the second order.

By a similar course of investigation, the cusps may be determined when the tangent is inclined to both the co-ordinate axes.

(289.) DEFINITION. An *isolated* point is a point whose co-ordinates satisfy the equation of a curve, while the point itself is entirely detached from every other in the curve.

PROPOSITION IX.—THEOREM.

For an isolated point, the first differential coefficient is equal to an imaginary constant.

For since, by supposition, the proposed point is entirely detached from every other point of the curve, there can be no tangent line corresponding to that point, and consequently the value of the first differential coefficient must be imaginary.

Ex. It is required to determine whether the curve represented by the equation

$$y^2=x(a+x)^2,$$

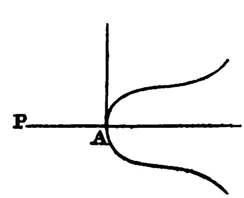
has an isolated point.

Extracting the square root, we find

$$y=\pm(a+x)\sqrt{x}.$$

Hence, when x is negative, y will be imaginary. If $x=0$, $y=0$, which shows that the curve passes through the origin A. For every positive value of x, y will have two real values, which shows that the curve has two branches extending indefinitely toward the right.

The equation is also satisfied by the values $x=-a$, $y=0$. Hence the point P, whose abscissa is $-a$, is detached from all others in the curve, and is called an isolated point. The form of the curve is such as exhibited in the annexed figure.



(290.) From the preceding propositions it will be seen that, in order to trace out a curve from its equation, we first dis-

[illegible] the most remarkable points by putting x and y successively equal to zero or infinity, and also the first and second differential coefficients equal to zero or infinity. Then, to trace the curve in the neighbourhood of the points thus determined, when they appear to present any peculiarity, we increase one of the co-ordinates by a very small quantity, and observe the effect upon the other co-ordinate. Having determined the singular points, and examined the course of the curve in their immediate vicinity, we can easily trace the remainder of the curve by assigning to x and y arbitrary values at pleasure.

INTEGRAL CALCULUS.

SECTION I.

INTEGRATION OF MONOMIAL DIFFERENTIALS — OF BINOMIAL DIFFERENTIALS — OF THE DIFFERENTIALS OF CIRCULAR ARCS.

ARTICLE (291.) THE Integral Calculus is the reverse of the Differential Calculus, its object being to determine the expression or function from which a given differential has been derived.

Thus we have found that the differential of x^2 is $2xdx$; therefore, if we have given $2xdx$, we know that it must have been derived from x^2, or x^2 plus a constant term.

(292.) The function from which the given differential has been derived, is called its *integral*. Hence, as we are not certain whether the integral has a constant quantity or not added to it, we annex a constant quantity represented by C, the value of which is to be determined from the nature of the problem.

(293.) Leibnitz considered the differentials of functions as indefinitely small differences, and the *sum* of these indefinitely small differences he regarded as making up the function; hence the letter S was placed before the differential to show that the sum was to be taken. As it was frequently required to place S before a compound expression, it was elongated into the sign $\int$, which, being placed before a differential, denotes that its integral is to be taken. Thus,

$$\int 2xdx = x^2 + C.$$

This sign $\int$ is still retained even by those who reject the philosophy of Leibnitz.

(294.) We have seen that the differential coefficient expresses the ratio of the rate of variation of the function to that of the independent variable. Hence, when we have given a certain differential to find its integral, it is to be understood

that we have given a certain quantity which varies uniformly, and the ratio of its rate of variation to another quantity depending on it and given quantities, to find the value of that quantity.

Thus, if we have given

$$du=3x^2dx,$$

to find its integral, we have given a quantity x which varies uniformly, and the ratio of its rate of variation to that of u, to find the value of u. And since

$$\int 3x^2dx=x^3,$$

we know that u equals x^3, or x^3+C.

Ex. There is a quantity x which increases uniformly, and the rate of its variation, compared with another quantity depending on it, is as 1 to ax^2; required the value of this quantity when $a=9$ and $x=10$.

Let u=the quantity required.

Then $\qquad dx : du :: 1 : ax^2.$

Hence $\qquad du=ax^2dx,$

and $\qquad \int du=\int ax^2dx,$

or $\qquad u=\dfrac{ax^3}{3}.$

Hence the number required is

$$\tfrac{9}{3}\times 10^3=3000.$$

(295.) We have seen (Art. 176) that the differential of the product of a variable multiplied by a constant, is equal to the constant multiplied by the differential of the variable. Hence we conclude that the integral of any differential multiplied by a constant quantity, is equal to the constant multiplied by the integral of the differential.

Thus, since the differential of ax is adx, it follows that

$$\int adx=ax=a\int dx.$$

Hence,

PROPOSITION I.—THEOREM.

If the expression to be integrated have a constant factor, this factor may be placed without the sign of integration.

Thus, $\qquad \int abx^2dx=ab\int x^2dx.$

(296.) We have seen (Art. 179) that the differential of a function composed of several terms, is equal to the sum or dif-

ference of the differentials taken separately. Hence the integral of a differential expression composed of several terms is equal to the sum or difference of the integrals taken separately. Thus, since the differential of

$$a^2x^2-2ax^3-x$$

is $$2a^2xdx-6ax^2dx-dx,$$

we conclude that $$\int(2a^2xdx-6ax^2dx-dx)$$

is $$a^2x^2+2ax^3-x.$$

Hence we derive

PROPOSITION II.—THEOREM.

The integral of the sum or difference of any number of differentials, is equal to the sum or difference of their respective integrals.

(297.) We have seen (Art. 177) that every constant quantity connected with the variable by the sign plus or minus will disappear in differentiation; that is, the differential of $u+C$ is the same as that of u. Consequently, the same differential may answer to several integral functions, differing from each other only in the value of the constant term. Hence

PROPOSITION III.—THEOREM.

In integrating, a constant term must always be added to the integral.

Thus, $$\int du=u+C.$$

(298.) We have found (Art. 186) that the differential of x^{m+1} is $$(m+1)x^m dx.$$

Hence $$x^m dx=\frac{dx^{m+1}}{m+1}=d\left(\frac{x^{m+1}}{m+1}\right).$$

Therefore $\frac{x^{m+1}}{m+1}$ is the function whose differential is $x^m dx$, or

$$\int x^m dx=\frac{x^{m+1}}{m+1}+C.$$

Hence

PROPOSITION IV.—THEOREM.

To find the integral of a monomial differential of the form $x^m dx$, *increase the exponent of the variable by unity, and then divide by the new exponent and by the differential of the variable.*

Ex. 1. The rate of variation of the independent variable x,

is to the rate of variation of a certain algebraic expression as 1 to $\frac{a}{b}x^2$; it is required to find that expression.

Ans. $\frac{ax^3}{3b}+C.$

Ex. 2. What is the integral of $\frac{x^2dx}{3}$?

Ans. $\frac{x^3}{9}+C.$

Ex. 3. What is the integral of $x^{\frac{1}{3}}dx$?

Ans. $\frac{3}{4}x^{\frac{4}{3}}+C.$

Ex. 4. What is the integral of $\frac{dx}{\sqrt{x}}$ or $x^{-\frac{1}{2}}dx$?

Ans. $2x^{\frac{1}{2}}+C.$

Ex. 5. What is the integral of $\frac{dx}{x^3}$ or $x^{-3}dx$?

Ans. $-\frac{x^{-2}}{2}$ or $-\frac{1}{2x^2}+C.$

Ex. 6. What is the integral of $ax^2dx+\frac{dx}{2\sqrt{x}}$?

Ans. $\frac{ax^3}{3}+x^{\frac{1}{2}}+C.$

Ex. 7. If the side of a square increases uniformly at the rate of $\frac{1}{10}$ of an inch per second, what is the area of the square when it is increasing at the rate of a square inch per second?

Ans.

(299.) There is one case in which the preceding rule fails. It is that in which the exponent m is equal to -1. For in this case we have, according to the rule,

$$\int x^{-1}dx=\frac{x^{-1+1}}{-1+1}=\frac{x^0}{0}=\frac{1}{0}=\infty,$$

which shows that the rule is inapplicable.

But $x^{-1}dx$ is the same as $\frac{dx}{x}$, and we know (Art. 215) that this expression was obtained by differentiating the logarithm of the denominator. Therefore

$$\int x^{-1}dx \text{ or } \int\frac{dx}{x}=\log. x+C.$$

Also, $$\int \frac{adx}{x} = a \log. x + C.$$

Hence we have

PROPOSITION V.—THEOREM.

If the numerator of a fraction is the product of a constant quantity by the differential of the denominator, its integral is the product of the constant by the Naperian logarithm of the denominator.

Ex. 1. What is the integral of $\frac{dx}{a+x}$?

Ans. $\log. (a+x)+C$.

Ex. 2. What is the integral of $\frac{2bxdx}{a+bx^2}$?

Ans. $\log. (a+bx^2)+C$.

Ex. 3. What is the integral of $\frac{adx}{bx}$?

Ans.

Ex. 4. What is the integral of $\frac{ax^2dx}{x^3}$?

Ans.

PROPOSITION VI.—THEOREM.

(300.) *Every polynomial of the form*

$$(a+bx+cx^2+, \text{etc.})^n dx,$$

in which n *is a positive whole number, may be integrated by raising the quantity within the parenthesis to the* n*th power, multiplying each term by* dx, *and then integrating each term separately.*

This is an obvious consequence of Proposition II.

Ex. 1. What is the integral of $(a+bx)^2dx$?

Expanding the quantity within the parenthesis, and multiplying each term by dx, we have

$$a^2dx+2abxdx+b^2x^2dx.$$

Integrating each term separately, we obtain

$$a^2x+abx^2+\frac{b^2x^3}{3}+C,$$

which is the integral sought.

Ex. 2. What is the integral of $(5+7x^2)^2dx$?

Ans.

Ex. 3. What is the integral of $(a+3x^2)^2dx$?

Ans.

(301.) We have seen (Art. 188) that any power of a polynomial may be differentiated by diminishing the exponent of the power by unity, and then multiplying by the primitive exponent and by the differential of the polynomial. Thus the differential of $(ax+x^2)^3$ is

$$3(ax+x^2)^2(adx+2xdx).$$

Hence we deduce

PROPOSITION VII.—THEOREM.

In order to integrate a compound expression consisting of any power of a polynomial multiplied by its differential, increase the exponent of the polynomial by unity, and then divide by the new exponent, and by the differential of the polynomial.

Ex. 1. What is the integral of $(a+3x^2)^2 6xdx$?

Ans. $\frac{(a+3x^2)^3}{3}+C.$

Ex. 2. What is the integral of $(2x^3-1)6x^2dx$?

Ans.

(302.) The preceding rule is equally applicable when the exponent of the polynomial is fractional.

Ex. 3. What is the integral of $(x+ax)^{\frac{1}{2}}(dx+adx)$?

Ans. $\frac{2}{3}(x+ax)^{\frac{3}{2}}+C.$

Ex. 4. What is the integral of $\frac{dx}{(1+x)^{\frac{1}{3}}}$?

Ans.

Ex. 5. What is the integral of $(ax^2+bx^3)^2(2ax+3bx^2)dx$?

Ans.

Ex. 6. What is the integral of $(ax+bx^2)^{\frac{1}{3}}(a+2bx)dx$?

Ans.

(303.) Any binomial differential of the form

$$du=(a+bx^n)^m x^{n-1}dx,$$

in which the exponent of the variable without the parenthesis

is one less than the exponent of the variable within, may be integrated in the same manner.

Let us put $\qquad y=a+bx^n.$

Then $\qquad dy=bnx^{n-1}dx,$

and $\qquad \frac{dy}{bn}=x^{n-1}dx.$

Therefore $\qquad du=y^m\frac{dy}{bn},$

and $\qquad u=\frac{y^{m+1}}{(m+1)bn}+C,$

or $\qquad u=\frac{(a+bx^n)^{m+1}}{(m+1)bn}+C.$

Hence we deduce

PROPOSITION VIII.—THEOREM.

To integrate a binomial differential when the exponent of the variable without the parenthesis is one less than that within, increase the exponent of the binomial by unity, and divide by the product of the new exponent, the coefficient, and the exponent of the variable within the parenthesis.

Ex. 1. What is the integral of $du=(a+3x^2)^3xdx$?

Let us put $\qquad y=a+3x^2;$

whence $\qquad dy=6xdx$, or $xdx=\frac{dy}{6}.$

Therefore $\qquad du=\frac{y^3dy}{6},$

and $\qquad u=\frac{y^4}{24}=\frac{(a+3x^2)^4}{24}+C.$

Ex. 2. What is the integral of $(a+bx^2)^{\frac{1}{2}}mxdx$?

Ans. $\frac{m}{3b}(a+bx^2)^{\frac{3}{2}}+C.$

Ex. 3. What is the integral of $\frac{xdx}{\sqrt{a^2+x^2}}$?

Ans. $\sqrt{a^2+x^2}+C.$

Ex. 4. What is the integral of $(a+bx^2)^{\frac{3}{2}}exdx$?

Ans. $\frac{e(a+bx^2)^{\frac{5}{2}}}{5b}+C$

Ex. 5. What is the integral of $(a^2+x^n)^2anx^{n-1}dx$?

Ans.

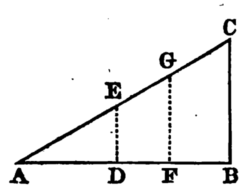
Ex. 6. If x increase uniformly at the rate of one inch per second, what is the form and value of the expression which is increasing at the rate of $\dfrac{1+x}{\sqrt{2x+x^2}}$ inches per second, when $x=10$ inches?

Ans.

(304.) To complete each integral as determined by the preceding rules, we have added a constant quantity C. While the value of this constant is unknown, the expression is called an *indefinite integral.* But in the application of the calculus to the solution of real problems, the complete value of the integral is determined by the conditions of the problem. We may determine the value of the constant, or make it disappear entirely from the integral, in the following manner. If we suppose the independent variable and the integral to begin to exist at the same instant, then when $x=0$, the integral $=0$, and consequently $C=0$.

Again, if we suppose the integral to begin to exist, or to have its origin when x becomes equal to a given quantity a, the value of C may then be determined.

When the value of the constant has been determined, and a particular value assigned to the independent variable, the value of the integral is then known, and is called a *definite integral.*



Ex. 1. Represent the base of the triangle ABC by x, and the perpendicular by nx, then the area of the triangle is $\frac{1}{2}nx^2$, whose differential is

$$nxdx.$$

If we take the integral of $nxdx$ according to Prop. IV., we obtain

$$\int nxdx=\tfrac{1}{2}nx^2+C,$$

which represents the area of the triangle ABC.

The constant is determined by observing that the base x, and the area of the triangle begin to exist at the same time; hence when $x=0$, the integral $=0$; that is,

$$\tfrac{1}{2}nx^2+C=0,$$

and consequently $$C=0.$$

Again, suppose we wish to obtain an expression for the area of the trapezoid EDFG, contained between the two perpendiculars DE, FG. We must first obtain the area of the triangle ADE. Suppose the variable x to be equal to AD, which we will represent by a, then the area of ADE will be expressed by

$$\tfrac{1}{2}na^2+C.$$

Next suppose the variable x to become equal to AF, which we will represent by b, then the area of AFG will be expressed by

$$\tfrac{1}{2}nb^2+C.$$

Subtracting the former expression from the latter, we obtain the area of the trapezoid EDFG,

$$\tfrac{1}{2}nb^2-\tfrac{1}{2}na^2.$$

(305.) Hence we find that *the constant* C *may be made to disappear by giving two successive values to the independent variable, and taking the difference between the two integrals corresponding to these values.*

When we take the excess of the value of an integral when the independent variable has become equal to b, above its value when it was only equal to a, we are said to integrate between the limits of $x=a$ and $x=b$.

This is indicated by the sign $\int_a^b$.

Thus, $$\int_a^b nxdx=\tfrac{1}{2}nb^2-\tfrac{1}{2}na^2=\frac{n(b^2-a^2)}{2}.$$

Ex. 2. Integrate $\int_a^b 2xdx$, and illustrate the case by a geometrical example.

Ans.

Ex. 3. Integrate $\int_a^b 3x^2dx$; illustrate the case by a geometrical example, and determine its numerical value when $a=4$ and $b=6$.

Ans.

Ex. 4. Integrate $\int_a^b \frac{\pi}{2}xdx$; illustrate the case by a geometrical example, and determine the value of the definite integral between the limits $a=2$ and $b=3$.

Ans.

Ex. 5. Integrate $\int_a^b \frac{\pi}{2}x^2dx$; illustrate the case by a geomet-

rical example, and determine the value of the definite integral between the limits $a=4$ and $b=6$.

Ans.

Ex. 6. What is the value of $\int_a^b 2(e+x)dx$, when $a=10, b=20$, and $e=4$? and illustrate the exercise by a geometrical figure.

Ans.

Ex. 7. What is the value of $\int_a^b 3(e+nx^2)^2 2nxdx$, when $a=4$, $b=6, e=4$, and $n=2$?

Ans.

Ex. 8. What is the value of $\int_a^b \frac{dx}{e+x}$, when $a=2, b=3$, and $e=4$?

Ans.

INTEGRATION BY SERIES.

(306.) If it is required to integrate an expression of the form

$$Xdx,$$

in which X is a function of x, it is often best to develop x into a series, and then, after multiplying by dx, to integrate each term separately. This is called integrating by series, since we thus obtain a series equal to the integral of the given expression, from which we may deduce the approximate value of the integral when the series is a converging one.

Ex. 1. It is required to integrate the expression $\frac{dx}{1+x}$.

By the binomial theorem, we find

$$\frac{1}{1+x} \text{ or } (1+x)^{-1}=1-x+x^2-x^3+x^4-, \text{ etc.}$$

Multiplying by dx, we have

$$\frac{dx}{1+x}=dx-xdx+x^2dx-x^3dx+x^4dx-, \text{ etc.};$$

and integrating each term separately, we obtain

$$\int\frac{dx}{1+x}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-, \text{ etc.}, +C.$$

Ex. 2. It is required to integrate the expression $\frac{dx}{1+x^2}$.

By the binomial theorem, we find

$$\frac{1}{1+x^2} \text{ or } (1+x^2)^{-1}=1-x^2+x^4-x^6+, \text{ etc.}$$

Whence $$\frac{dx}{1+x^2}=dx-x^2dx+x^4dx-x^6dx+, \text{ etc.},$$

and $$\int\frac{dx}{1+x^2}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+, \text{ etc.}, +C.$$

Ex. 3. What is the integral of $\frac{dx}{a-x}$?

Ans.

Ex. 4. What is the integral of $\frac{dx}{(a-x)^2}$?

Ans.

Ex. 5. What is the integral of $\frac{dx}{\sqrt{1-x^2}}$?

Ans.

INTEGRATION OF THE DIFFERENTIALS OF CIRCULAR ARCS.

(307.) We have found in Art. 227, that if z designates an arc, and y its sine, the radius of the circle being unity,

$$dz=\frac{dy}{\sqrt{1-y^2}}.$$

Hence $$\int\frac{dy}{\sqrt{1-y^2}}=z+C.$$

If the arc be estimated from the beginning of the first quadrant, the sine will be 0 when the arc is 0, and consequently C equals zero.

Therefore the entire integral is

$$\int\frac{dy}{\sqrt{1-y^2}}=\text{the arc of which } y \text{ is the sine.}$$

If it were required to integrate an expression of the form

$$dz=\frac{dy}{\sqrt{a^2-y^2}}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $$v=\frac{y}{a} \text{ or } y=av.$$

Then $$dy=adv, \text{ and } \sqrt{a^2-y^2}=a\sqrt{1-v^2}.$$

Substituting these values in equation (1), we have

$$dz=\frac{dv}{\sqrt{1-v^2}}.$$

Hence $\qquad z=$the arc whose sine is v,

or $\qquad z=$the arc whose sine is $\frac{y}{a}$.

Ex. Integrate the expression

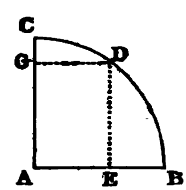
$$dz=\frac{dy}{\sqrt{4-y^2}}.$$

Ans. $z=$the arc whose sine is $\frac{1}{2}y$.

(308.) We have found in Art. 227, that if z designates an arc, and y' its cosine, the radius of the circle being unity,

$$dz=\frac{-dy'}{\sqrt{1-y'^2}}.$$

Hence $$\int\frac{-dy'}{\sqrt{1-y'^2}}=z+C.$$



To determine the constant C, we see that if the arc be estimated from B, the beginning of the first quadrant, the cosine becomes 0 when the arc becomes a quadrant, which is represented by $\frac{1}{2}\pi$; hence the first member of the equation becomes equal to $\frac{1}{2}\pi$ when $y'=0$. But under this supposition the arc whose cosine is 0 becomes $\frac{1}{2}\pi$; hence C=0, and the entire integral is

$$\int\frac{-dy'}{\sqrt{1-y'^2}}=\text{the arc whose cosine is } y'.$$

If it were required to integrate an expression of the form

$$dz=\frac{-dy'}{\sqrt{a^2-y'^2}},$$

it may be done by the aid of an auxiliary variable, as in Art. 307, and we shall find

$$z=\text{the arc whose cosine is } \frac{y'}{a}.$$

(309.) We have found in Art. 227, that if z designates an arc, and t its tangent,

$$dz=\frac{dt}{1+t^2}.$$

Hence $$\int \frac{dt}{1+t^2}=z+C.$$

If the arc is estimated from the beginning of the first quadrant, we shall have

$$z=0 \text{ when } \int \frac{dt}{1+t^2}=0\text{; hence } C=0,$$

and the entire integral is

$$\int \frac{dt}{1+t^2}=\text{the arc of which } t \text{ is the tangent.}$$

If it were required to integrate an expression of the form

$$dz=\frac{dt}{a^2+t^2}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $$v=\frac{t}{a} \text{ or } t=av\text{;}$$

then $$dt=adv.$$

Substituting in equation (1), we have

$$dz=\frac{adv}{a^2+a^2v^2}=\frac{1}{a}\left(\frac{dv}{1+v^2}\right).$$

Hence $$z=\frac{1}{a}\times \text{arc whose tangent is } v,$$

or $$z=\frac{1}{a}\times \text{arc whose tangent is } \frac{t}{a}.$$

(310.) We have found, Art. 227, that if z designates an arc, and x its versed sine,

$$dz=\frac{dx}{\sqrt{2x-x^2}}.$$

Hence $$\int \frac{dx}{\sqrt{2x-x^2}}=z+C.$$

If the arc is estimated from the beginning of the first quadrant, we shall have

$$C=0,$$

and the entire integral is

$$\int \frac{dx}{\sqrt{2x-x^2}}=\text{the arc of which } x \text{ is the versed sine.}$$

If it were required to integrate an expression of the form

$$dz=\frac{dx}{\sqrt{2ax-x^2}}, \qquad (1)$$

it may be done by the aid of an auxiliary variable.

Assume $\qquad v=\frac{x}{a}$ or $x=av$;

then $\qquad dx=adv.$

Substituting in equation (1), we have

$$dz=\frac{adv}{\sqrt{2a^2v-a^2v^2}}=\frac{dv}{\sqrt{2v-v^2}}.$$

Hence $\qquad z=$the arc whose versed sine is v,

or $\qquad z=$the arc whose versed sine is $\frac{x}{a}$.

INTEGRATION OF BINOMIAL DIFFERENTIALS.

PROPOSITION IX.—THEOREM.

(311.) *Every binomial differential can be reduced to the form*

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx,$$

in which the exponents m *and* n *are whole numbers, and* n *is positive.*

1st. For if m and n were fractional, and the binomial were of the form

$$x^{\frac{1}{3}}(a+bx^{\frac{1}{2}})^{\frac{p}{q}}dx,$$

we may substitute for x another variable, with an exponent equal to the least common multiple of the denominators of the given exponents, by which means the proposed binomial will be transformed into one in which the exponents of the variable are whole numbers.

Thus, if we make $x=z^6$, we find

$$x^{\frac{1}{3}}(a+bx^{\frac{1}{2}})^{\frac{p}{q}}dx=6z^7(a+bz^3)^{\frac{p}{q}}dz,$$

in which the exponents of z are whole numbers.

2d. If n were negative, or the expression were of the form

$$x^{m-1}(a+bx^{-n})^{\frac{p}{q}}dx$$

we may put $x=\frac{1}{z}$, in which case we shall obtain

$$-z^{-m-1}(a+bz^n)^{\frac{p}{q}}dz,$$

in which the exponent of the variable within the parenthesis is positive.

3d. If the variable x were found in both terms of the binomial, and the expression were of the form

$$x^{m-1}(ax^r+bx^n)^{\frac{p}{q}}dx,$$

we may divide the binomial within the parenthesis by x^r, and multiply the factor without the parenthesis by $x^{\frac{pr}{q}}$, and we shall obtain

$$x^{m+\frac{pr}{q}-1}(a+bx^{n-r})^{\frac{p}{q}}dx,$$

in which but one of the terms within the parenthesis contains the variable x.

PROPOSITION X.—THEOREM.

(312.) *Every binomial differential, in which the exponent of the parenthesis is a whole number and positive, can be integrated by raising the quantity within the parenthesis to the proposed power, multiplying each term by the factor without the parenthesis, and then integrating each term separately.*

This results directly from Proposition II.

Ex. 1. Integrate the expression

$$du=x^2(a+bx^3)^2dx.$$

Expanding the binomial, we obtain

$$du=a^2x^2dx+2abx^5dx+b^2x^8dx.$$

And integrating each term separately, we find

$$u=\frac{a^2x^3}{3}+\frac{abx^6}{3}+\frac{b^2x^9}{9}+C.$$

Ex. 2. Integrate the expression

$$du=x^3(a+bx^2)^3dx.$$

$$Ans.\quad u=\frac{a^3x^4}{4}+\frac{3a^2bx^6}{6}+\frac{3ab^2x^8}{8}+\frac{b^3x^{10}}{10}+C.$$

Ex. 3. Integrate the expression

$$du=x^4(a+bx^3)^3dx.$$

Ans.

Ex. 4. Integrate the expression

$$du=x^5(a+b^2x^4)^3dx.$$

Ans.

PROPOSITION XI.—THEOREM.

(313.) *Every binomial differential can be integrated, when the exponent of the variable without the parenthesis, increased by unity, is exactly divisible by the exponent of the variable within.*

For this purpose, we substitute for the binomial within the parenthesis, a new variable having an exponent equal to the denominator of the exponent of the parenthesis.

Let us assume $\qquad a+bx^n=z^q.$

Then $\qquad (a+bx^n)^{\frac{p}{q}}=z^p. \qquad (1)$

Also, $\qquad x^n=\frac{z^q-a}{b},$

and $\qquad x^m=\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}},$

and, by differentiating,

$$mx^{m-1}dx=\frac{mq}{nb}z^{q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz. \qquad (2)$$

Multiplying together equations (1) and (2), and dividing by m, we obtain

$$x^{m-1}(a+bx^n)^{\frac{p}{q}}dx=\frac{q}{nb}z^{p+q-1}\left(\frac{z^q-a}{b}\right)^{\frac{m}{n}-1}dz,$$

which, according to Prop. X., can be integrated when $\frac{m}{n}$ is a whole number and positive. If $\frac{m}{n}$ is negative, we may, by Formula D, Art. 323, increase the exponent until it becomes positive.

Ex. 1. Integrate the expression

$$du=x^3(a+bx^2)^{\frac{3}{2}}dx.$$

Assume $\qquad a+bx^2=z^2.$

Then $\qquad (a+bx^2)^{\frac{3}{2}}=z^3. \qquad (1)$

Also, $\qquad x^2=\frac{z^2-a}{b} \qquad (2)$

and $$xdx=\frac{zdz}{b}. \qquad (3)$$

Multiplying together equations (1), (2), and (3), we obtain

$$du=x^3(a+bx^2)^{\frac{3}{2}}dx=z^4.\frac{z^2-a}{b^2}dz.$$

Hence $$u=\frac{z^7}{7b^2}-\frac{az^5}{5b^2}+C.$$

Replacing the value of z, we find

$$u=\frac{(a+bx^2)^{\frac{7}{2}}}{7b^2}-\frac{a(a+bx^2)^{\frac{5}{2}}}{5b^2}+C.$$

Ex. 2. Integrate the expression

$$du=x^5(a+bx^2)^{\frac{1}{2}}dx.$$

Assume $$a+bx^2=z^2.$$

Then $$(a+bx^2)^{\frac{1}{2}}=z. \qquad (1)$$

Also, $$x^2=\frac{z^2-a}{b} \qquad (2)$$

and $$xdx=\frac{zdz}{b}. \qquad (3)$$

Hence $$du=\left(\frac{z^2-a}{b}\right)^2\frac{z^2dz}{b},$$

or $$du=\frac{z^6dz-2az^4dz+a^2z^2dz}{b^3},$$

and $$u=\frac{z^7}{7b^3}-\frac{2az^5}{5b^3}+\frac{a^2z^3}{3b^3}+C.$$

Restoring the value of z, we find

$$u=\frac{(a+bx^2)^{\frac{7}{2}}}{7b^3}-\frac{2a(a+bx^2)^{\frac{5}{2}}}{5b^3}+\frac{a^2(a+bx^2)^{\frac{3}{2}}}{3b^3}+C.$$

Ex. 3. Integrate the expression

$$du=x^5(a+bx^2)^{\frac{2}{3}}dx.$$

$$Ans.\ u=\frac{3(a+bx^2)^{\frac{11}{3}}}{22b^3}-\frac{3a(a+bx^2)^{\frac{8}{3}}}{8b^3}+\frac{3a^2(a+bx^2)^{\frac{5}{3}}}{10b^3}+C.$$

Ex. 4. Integrate the expression

$$du=x^3(a-x^2)^{-\frac{1}{2}}dx.$$

If we put $a-x^2=z^2$, we find

$$du=-(a-z^2)dz.$$

Whence
$$u=-az+\frac{z^3}{3}+\text{C},$$

or
$$u=-a(a-x^2)^{\frac{1}{2}}+\frac{(a-x^2)^{\frac{3}{2}}}{3}+\text{C}.$$

Ex. 5. Integrate the expression

$$du=x^5(a^2+x^2)^{-1}dx.$$

If we put $z=a^2+x^2$, we find

$$du=\frac{zdz}{2}-a^2dz+\frac{a^4dz}{2z},$$

and
$$u=\frac{z^2}{4}-a^2z+\frac{a^4}{2}\log. z+\text{C}.$$

or
$$u=\left(\frac{a^2+x^2}{2}\right)^2-a^2(a^2+x^2)+\frac{a^4}{2}\log. (a^2+x^2)+\text{C}.$$

PROPOSITION XII.—THEOREM.

(314.) *Every binomial differential can be integrated, when the exponent of the variable without the parenthesis, augmented by unity, and divided by the exponent of the variable within the parenthesis, plus the exponent of the parenthesis, is a whole number.*

The binomial $x^{m-1}(a+bx^n)^{\frac{p}{q}}dx$, may be written

$$x^{m-1}\left[\left(\frac{a}{x^n}+b\right)x^n\right]^{\frac{p}{q}}dx,$$

or
$$x^{m-1}(ax^{-n}+b)^{\frac{p}{q}}x^{\frac{np}{q}}dx,$$

which equals
$$x^{m+\frac{np}{q}-1}(ax^{-n}+b)^{\frac{p}{q}}dx,$$

which, according to the preceding Proposition, can be integrated when

$$\frac{m+\dfrac{np}{q}}{n}\text{ is a whole number,}$$

or
$$\left(\frac{m}{n}+\frac{p}{q}\right)\text{ is a whole number.}$$

Ex. 1. Integrate the expression

$$du=a(1+x^2)^{-\frac{5}{2}}dx.$$

Put $$v^2x^2=1+x^2.$$

Then $$(1+x^2)^{-\frac{3}{2}}=v^{-3}x^{-3}. \qquad (1)$$

Also, $$x^2=\frac{1}{v^2-1}.$$

Whence $$dx=\frac{-vdv}{x(v^2-1)^2}, \qquad (2)$$

and $$1=x^4(v^2-1)^2. \qquad (3)$$

Multiplying together equations (1), (2), and (3), we have

$$du=a(1+x^2)^{-\frac{3}{2}}dx=-\frac{adv}{v^2}.$$

Whence $$u=\frac{a}{v}=\frac{ax}{\sqrt{1+x^2}}+\mathrm{C}.$$

Ex. 2. Integrate the expression

$$du=x^{-4}(1-x^2)^{-\frac{1}{2}}dx.$$

Put $$v^2x^2=1-x^2.$$

Then $$x^{-2}=v^2+1,$$

and $$x^{-4}=(v^2+1)^2. \qquad (1)$$

Also, $$x=(v^2+1)^{-\frac{1}{2}}.$$

Whence $$dx=-(v^2+1)^{-\frac{3}{2}}vdv. \qquad (2)$$

Also, $$(1-x^2)^{-\frac{1}{2}}=\frac{1}{vx}=\frac{(v^2+1)^{\frac{1}{2}}}{v}. \qquad (3)$$

Multiplying together equations (1), (2), and (3), we have

$$du=x^{-4}(1-x^2)^{-\frac{1}{2}}dx=-(v^2+1)dv.$$

Whence $$u=-\frac{v^3}{3}-v=-\frac{v(v^2+3)}{3}+\mathrm{C},$$

or $$u=-\frac{1+2x^2}{3x^3}\sqrt{1-x^2}+\mathrm{C}.$$

Ex. 3. Integrate the expression

$$d\mathrm{X}_0=dx(a^2+x^2)^{-\frac{1}{2}}, \text{ or } \frac{dx}{\sqrt{a^2+x^2}}.$$

Put $$v=x+\sqrt{a^2+x^2}.$$

Then $$dv=dx+\frac{xdx}{\sqrt{a^2+x^2}}=\frac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}dx.$$

Therefore $$\frac{dv}{v}=\frac{dx}{\sqrt{a^2+x^2}}.$$

Consequently we have

$$X_1=\int\frac{dx}{\sqrt{a^2+x^2}}=\int\frac{dv}{v}=\log.\ v=\log.\ [x+\sqrt{a^2+x^2}].$$

Ex. 4. Integrate the expression

$$dX_2=\frac{x^2dx}{\sqrt{a^2+x^2}}.$$

Put $\qquad v=(a^2x^2+x^4)^{\frac{1}{2}}.$

Whence $\qquad dv=\dfrac{a^2xdx+2x^3dx}{(a^2x^2+x^4)^{\frac{1}{2}}},$

or $\qquad dv=\dfrac{a^2dx}{(a^2+x^2)^{\frac{1}{2}}}+\dfrac{2x^2dx}{(a^2+x^2)^{\frac{1}{2}}},$

or $\qquad dv=a^2dX_1+2dX_2,$

where X_1 has the same value as in Ex. 3.

Whence $\qquad dX_2=\dfrac{dv}{2}-\dfrac{a^2dX_1}{2},$

and $\qquad X_2=\dfrac{v}{2}-\dfrac{a^2X_1}{2},$

or $\qquad X_2=\dfrac{x}{2}(a^2+x^2)^{\frac{1}{2}}-\dfrac{a^2}{2}X_1.$

(315.) When a binomial differential can not be integrated by either of the preceding methods, its integral may be made to depend upon the integral of another differential of a simpler form. This is effected by resolving the binomial into two parts, one of which has a known integral.

We have seen, Art. 180, that

$$d(uv)=udv+vdu;$$

whence, by integrating,

$$uv=\int udv+\int vdu,$$

and consequently,

$$\int udv=uv-\int vdu, \qquad (1)$$

a formula which reduces the integration of udv to that of vdu, and which is known by the name of *integration by parts*.

For greater convenience, we shall represent the binomial differential by $x^m(a+bx^n)^pdx$, where p is supposed to be a fraction, but m and n are whole numbers.

PROPOSITION XIII.—THEOREM.

(316.) *The integral of any differential of the form*

$$x^m(a+bx^n)^p dx,$$

may be made to depend upon the integral of another differential of the same form, but in which the exponent of the variable without the parenthesis is diminished by the exponent of the variable within the parenthesis.

Let us put $$v=(a+bx^n)^s,$$
where s is an exponent to which any value may be assigned, as may be found most convenient.

Differentiating, we find

$$dv=bnsx^{n-1}(a+bx^n)^{s-1}dx.$$

If then we assume

$$udv=x^m(a+bx^n)^p dx,$$

we must have $$u=\frac{x^{m-n+1}(a+bx^n)^{p-s+1}}{bns},$$

and, by differentiating,

$$du=\frac{(m-n+1)x^{m-n}(a+bx^n)^{p-s+1}}{bns}dx+\frac{(p-s+1)x^m(a+bx^n)^{p-s}}{s}dx.$$

But $$(a+bx^n)^{p-s+1}=(a+bx^n)(a+bx^n)^{p-s}$$
$$=a(a+bx^n)^{p-s}+bx^n(a+bx^n)^{p-s}.$$

Hence

$$du=\left[\frac{a(m-n+1)x^{m-n}}{bns}+\frac{(m+1+np-ns)x^m}{ns}\right](a+bx^n)^{p-s}dx.$$

Let the value of s be taken such that

$$m+1+np-ns=0;$$

that is, $$s=\frac{m+1}{n}+p,$$

and we shall have

$$du=\frac{a(m-n+1)x^{m-n}(a+bx^n)^{p-s}}{b(np+m+1)}dx.$$

Substituting the values of u, v, du, and dv, here given, in formula (1), Art. 315, we obtain

FORMULA A.

$$\int x^m(a+bx^n)^p dx=\frac{x^{m-n+1}(a+bx^n)^{p+1}-a(m-n+1)\int x^{m-n}(a+bx^n)^p dx}{b(np+m+1)},$$

by which the integral of the proposed differential is made to depend upon the integral of

$$x^{m-n}(a+bx^n)^p dx,$$

in which the exponent of the factor x^m without the parenthesis, is diminished by that of x^n, the variable within the parenthesis; and by a similar process we should find

$$\int x^{m-n}(a+bx^n)^p dx,$$

to depend on $\int x^{m-2n}(a+bx^n)^p dx$;

and by continuing this process, the exponent of the factor without the parenthesis may be diminished until it is less than n.

(317.) We have frequent occasion to integrate binomial differentials of the form

$$dX_m = \frac{x^m dx}{\sqrt{a^2-x^2}}.$$

This may be done by Formula A, Art. 316, by substituting

a^2 for a,
-1 for b,
2 for n,
$-\frac{1}{2}$ for p.

Whence we obtain

FORMULA a.

$$X_m = \int \frac{x^m dx}{\sqrt{a^2-x^2}} = \frac{(m-1)a^2}{m}\int \frac{x^{m-2}dx}{\sqrt{a^2-x^2}} - \frac{x^{m-1}}{m}\sqrt{a^2-x^2}.$$

Ex. 1. Integrate the expression

$$dX_0 = \frac{adx}{\sqrt{a^2-x^2}}.$$

We have found in Art. 226, that $\frac{Rdy}{\sqrt{R^2-y^2}}$ is the differential of an arc of a circle of which R is the radius and y is the sine. Hence dX_0 is the differential of a circular arc, and X_0 is the arc of a circle of which a is the radius and x is the sine.

Ex. 2. Integrate the expression

$$dX_2 = \frac{x^2 dx}{\sqrt{a^2-x^2}}.$$

Make m, in Formula a, equal to 2, and the formula reduces to

$$X_2=\frac{a^2}{2}X_0-\frac{x}{2}\sqrt{a^2-x^2},$$

where X_0 has the same value as in Ex. 1.

Ex. 3. Integrate the expression

$$dX_4=\frac{x^4dx}{\sqrt{a^2-x^2}}.$$

Make m, in Formula a, equal to 4, and the formula reduces to

$$X_4=\frac{3a^2}{4}X_2-\frac{x^3}{4}\sqrt{a^2-x^2},$$

where X_2 has the same value as in Ex. 2.

Ex. 4. Integrate the expression

$$dX_6=\frac{x^6dx}{\sqrt{a^2-x^2}}.$$

Make m, in Formula a, equal to 6, and the formula reduces to

$$X_6=\frac{5a^2}{6}X_4-\frac{x^5}{6}\sqrt{a^2-x^2},$$

where X_4 has the same value as in Ex. 3.

Ex. 5. Integrate the expression

$$dX_8=\frac{x^8dx}{\sqrt{a^2-x^2}}.$$

Make m, in Formula a, equal to 8, and the formula reduces to

$$X_8=\frac{7a^2}{8}X_6-\frac{x^7}{8}\sqrt{a^2-x^2},$$

where X_6 has the same value as in Ex. 4.

(318.) Formula a reduces the binomial differential

$$\int\frac{x^m dx}{\sqrt{a^2-x^2}},$$

to that of $\int\frac{x^{m-2}dx}{\sqrt{a^2-x^2}};$

and, in a similar manner, this is found to depend upon

$$\int\frac{x^{m-4}dx}{\sqrt{a^2-x^2}},$$

and so on; so that after $\frac{m}{2}$ operations, when m is an even number, the integral is found to depend upon

$$\int \frac{dx}{\sqrt{a^2-x^2}},$$

which represents a circular arc whose sine is $\frac{x}{a}$.

In a similar manner is derived

FORMULA *b*.

$$X_m=\int \frac{x^m dx}{\sqrt{a^2+x^2}}=\frac{x^{m-1}}{m}\sqrt{a^2+x^2}-\frac{(m-1)a^2}{m}\int \frac{x^{m-2}dx}{\sqrt{a^2+x^2}}.$$

Ex. Integrate the expression

$$dX_4=\frac{x^4 dx}{\sqrt{a^2+x^2}}.$$

Make $m=4$, in Formula *b*, and it reduces to

$$X_4=\frac{x^3}{4}\sqrt{a^2+x^2}-\frac{3a^2}{4}\int \frac{x^2 dx}{\sqrt{a^2+x^2}}.$$

The integral of $\frac{x^2 dx}{\sqrt{a^2+x^2}}$ has been given in Art. 314, Ex. 4.

(319.) The binomial

$$dX_m=\frac{x^m dx}{\sqrt{2ax-x^2}},$$

may be integrated by means of Formula A, Art. 316, by making the proper substitutions. It may, however, be integrated by an independent process as follows:

Let us put $v=x^{m-1}\sqrt{2ax-x^2}$, or $(2ax^{2m-1}-x^{2m})^{\frac{1}{2}}$

Differentiating, we find

$$dv=\frac{a(2m-1)x^{2m-2}dx-mx^{2m-1}dx}{(2ax^{2m-1}-x^{2m})^{\frac{1}{2}}},$$

$$=\frac{a(2m-1)x^{m-1}dx}{(2ax-x^2)^{\frac{1}{2}}}-\frac{mx^m dx}{(2ax-x^2)^{\frac{1}{2}}}.$$

But this last term is equal to $m.dX_m$; hence

$$dX_m=\frac{a(2m-1)x^{m-1}dx}{m(2ax-x^2)^{\frac{1}{2}}}-\frac{dv}{m},$$

and, by integrating we obtain

FORMULA c.

$$\int\frac{x^m dx}{\sqrt{2ax-x^2}}=\frac{a(2m-1)}{m}\int\frac{x^{m-1}dx}{\sqrt{2ax-x^2}}-\frac{x^{m-1}}{m}\sqrt{2ax-x^2}$$

which diminishes the exponent of the variable without the parenthesis by unity.

Ex. 1. Integrate the expression

$$dX_0=\frac{adx}{\sqrt{2ax-x^2}}.$$

We have found in Art. 226, that $\frac{Rdx}{\sqrt{2Rx-x^2}}$ is the differential of an arc of a circle of which R is the radius and x is the versed sine. Hence dX_0 is the differential of a circular arc, and X_0 is the arc of a circle of which a is the radius and x is the versed sine.

Ex. 2. Integrate the expression

$$dX_1=\frac{xdx}{\sqrt{2ax-x^2}}.$$

Make m, in Formula c, equal to unity, and the formula reduces to

$$X_1=X_0-\sqrt{2ax-x^2},$$

where X_0 has the same value as in Ex. 1.

Ex. 3. Integrate the expression

$$dX_2=\frac{x^2dx}{\sqrt{2ax-x^2}}.$$

Make m, in Formula c, equal to 2, and the formula reduces to

$$X_2=\frac{3a}{2}X_1-\frac{x}{2}\sqrt{2ax-x^2},$$

where X_1 has the same value as in Ex. 2.

Ex. 4. Integrate the expression

$$dX_3=\frac{x^3dx}{\sqrt{2ax-x^2}}.$$

Make m, in Formula c, equal to 3, and the formula reduces to

$$X_3=\frac{5a}{3}X_2-\frac{x^2}{3}\sqrt{2ax-x^2},$$

where X_2 has the same value as in Ex. 3.

Ex. 5. Integrate the expression

$$dX_4=\frac{x^4dx}{\sqrt{2ax-x^2}}.$$

Make m, in Formula c, equal to 4, and the formula reduces to

$$X_4=\frac{7a}{4}X_3-\frac{x^3}{4}\sqrt{2ax-x^2},$$

where X_3 has the same value as in Ex. 4.

(320.) Formula c reduces the differential binomial

$$\int\frac{x^mdx}{\sqrt{2ax-x^2}},$$

to that of

$$\int\frac{x^{m-1}dx}{\sqrt{2ax-x^2}};$$

and, in a similar manner, this will be found to depend upon

$$\int\frac{x^{m-2}dx}{\sqrt{2ax-x^2}},$$

and so on; so that after m operations, when m is a whole number, the integral is found to depend upon

$$\int\frac{dx}{\sqrt{2ax-x^2}},$$

which represents the arc whose versed sine is $\frac{x}{a}$.

Ex. 6. It is required to find the integral of

$$\frac{x^{\frac{3}{2}}dx}{\sqrt{2ax-x^2}}.$$

Substituting, in Formula c, $\frac{3}{2}$ for m, we obtain

$$\int\frac{x^{\frac{3}{2}}dx}{\sqrt{2ax-x^2}}=\frac{4a}{3}\int\frac{x^{\frac{1}{2}}dx}{\sqrt{2ax-x^2}}-\frac{2x^{\frac{1}{2}}}{3}\sqrt{2ax-x^2}.$$

But

$$\frac{x^{\frac{1}{2}}dx}{\sqrt{2ax-x^2}}=\frac{dx}{\sqrt{2a-x}}.$$

Also

$$\int\frac{dx}{\sqrt{2a-x}}=-2\sqrt{2a-x}.$$

Hence

$$\int\frac{x^{\frac{3}{2}}dx}{\sqrt{2ax-x^2}}=-\frac{8a}{3}\sqrt{2a-x}-\frac{2x}{3}\sqrt{2a-x}.$$

(321.) Formula A will only diminish the exponent m when

m is positive; but we may easily deduce from this formula another which will diminish the exponent when it is negative. For this purpose, multiply Formula A by the denominator $b(np+m+1)$, and transposing the term which does not contain the sign of integration, we obtain

FORMULA B.

$$\int x^{m-n}(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1} - b(np+m+1)\int x^m(a+bx^n)^p dx}{a(m-n+1)}.$$

Ex. 1. Find the integral of

$$\frac{dx}{x^2(1+x^3)^{\frac{1}{3}}}, \text{ or } x^{-2}(1+x^3)^{-\frac{1}{3}}dx.$$

Substituting, in Formula B, -2 for $m-n$,
1 for a,
1 for b,
3 for n,
and $-\frac{1}{3}$ for p, we obtain

$$\int x^{-2}(1+x^3)^{-\frac{1}{3}}dx = -x^{-1}(1+x^3)^{\frac{2}{3}} + \int x(1+x^3)^{-\frac{1}{3}}dx.$$

Ex. 2. Find the integral of

$$\frac{dx}{x^2(2-x^2)^{\frac{3}{2}}}, \text{ or } x^{-2}(2-x^2)^{-\frac{3}{2}}dx.$$

Substituting, in Formula B, -2 for $m-n$,
2 for a,
-1 for b,
2 for n,
and $-\frac{3}{2}$ for p, we obtain

$$\int x^{-2}(2-x^2)^{-\frac{3}{2}}dx = -\frac{x^{-1}(2-x^2)^{-\frac{1}{2}}}{2} + \int(2-x^2)^{-\frac{3}{2}}dx.$$

PROPOSITION XIV.—THEOREM.

(322.) *The integral of any differential of the form*

$$x^m(a+bx^n)^p dx,$$

may be made to depend upon the integral of another differential of the same form, but in which the exponent of the parenthesis is diminished by unity.

Let us put $\qquad v=x^n,$

where s is an exponent to which any value may be assigned, as may be found most convenient.

Differentiating, we find

$$dv = sx^{s-1}dx.$$

If then we assume $udv = x^m(a+bx^n)^p dx$,

we must have $$u = \frac{x^{m-s+1}}{s}(a+bx^n)^p;$$

and, by differentiating,

$$du = \frac{(m-s+1)}{s}x^{m-s}(a+bx^n)^p dx + \frac{bnp}{s}x^{m-s+n}(a+bx^n)^{p-1}dx.$$

But $$(a+bx^n)^p = (a+bx^n)(a+bx^n)^{p-1}.$$

Hence

$$du = \frac{a(m-s+1)+b(m-s+1+np)x^n}{s}x^{m-s}(a+bx^n)^{p-1}dx.$$

Let the value of s be taken such that

$$m-s+1+np=0;$$

that is, $$s=m+1+np,$$

we shall have $$du = \frac{-anpx^{m-s}(a+bx^n)^{p-1}dx}{np+m+1}.$$

Substituting the values of u, v, du, and dv here given in formula (1), Art. 315, we obtain

FORMULA C.

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p + anp\int x^m(a+bx^n)^{p-1}dx}{np+m+1},$$

by which the value of the required integral is made to depend upon another having the exponent of the binomial less by unity. The value of this new integral may, by the same formula, be made to depend upon that of an integral in which the exponent of the binomial is still further diminished; and so on until the exponent of the binomial is reduced to a fraction less than unity.

Ex. 1. Find the integral of the expression

$$dx\sqrt{a^2+x^2}.$$

We may diminish the exponent of the binomial by unity by substituting, in Formula C, 0 for m,
a^2 for a,
1 for b,
2 for n,
$\frac{1}{2}$ for p, and we obtain

$$\int dx\sqrt{a^2+x^2}=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\int\frac{dx}{\sqrt{a^2+x^2}}.$$

But by Ex. 3, Art. 314,

$$\int\frac{dx}{\sqrt{a^2+x^2}}=\log.[x+\sqrt{a^2+x^2}].$$

Hence $\int dx\sqrt{a^2+x^2}=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log.[x+\sqrt{a^2+x^2}].$

Ex. 2. Find the integral of the expression

$$dx\sqrt{x^2-a^2}.$$

$$\textit{Ans. } \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\log.[x+\sqrt{x^2-a^2}].$$

(323.) Formula C will only diminish the exponent of the parenthesis when the exponent is positive; but we may easily deduce from this formula another which will diminish the exponent when it is negative. For this purpose, multiply Formula C by the denominator $np+m+1$, and transposing the term which does not contain the sign of integration, we obtain

FORMULA D.

$$\int x^m(a+bx^n)^{p-1}dx=\frac{-x^{m+1}(a+bx^n)^p+(np+m+1)\int x^m(a+bx^n)^p dx}{anp}$$

Ex. 1. Find the integral of

$$(2-x^2)^{-\frac{3}{2}}dx.$$

Substituting, in Formula D, 0 for m,
2 for a,
-1 for b,
2 for n,
and $-\frac{3}{2}$ for $p-1$, we obtain

$$\int(2-x^2)^{-\frac{3}{2}}dx=\frac{x}{2}(2-x^2)^{-\frac{1}{2}}, \text{ or } \frac{x}{2\sqrt{2-x^2}}.$$

Ex. 2. Find the integral of

$$\frac{xdx}{(1+x^2)^{\frac{3}{2}}}, \text{ or } x(1+x^2)^{-\frac{3}{2}}dx.$$

Substituting, in Formula D, 1 for m,
1 for a,

1 for b,

3 for n,

and $-\frac{1}{3}$ for $p-1$, we obtain

$$\int x(1+x^3)^{-\frac{1}{3}}dx=-\frac{x^2}{2}(1+x^3)^{\frac{2}{3}}+2\int x(1+x^3)^{\frac{2}{3}}dx,$$

where $x(1+x^3)^{\frac{2}{3}}dx$ may be developed in a series, and each term integrated separately.

SECTION II.

APPLICATIONS OF THE INTEGRAL CALCULUS—RECTIFICATION OF CURVES.

(324.) To *rectify* a curve is to obtain a straight line equal to an arc of the curve. When an expression for the length of a curve can be found in a finite number of algebraic terms, the curve is said to be *rectifiable.*

We have found (Art. 251) that the differential of an arc of a curve, referred to rectangular co-ordinates, is

$$dz=\sqrt{dx^2+dy^2};$$

whence
$$z=\int\sqrt{dx^2+dy^2}, \qquad (1)$$

which is a general expression for the length of an indefinite portion of any curve referred to rectangular co-ordinates.

In order, then, to rectify a curve given by its equation, *we differentiate its equation, and deduce from it the value of* dx *or* dy, *which we substitute in expression* (1). *The radical will then contain but one variable, which being the differential of the arc, its integral will be the length of the arc itself.*

Ex. 1. It is required to find the length of an arc of the semicubical parabola whose equation is

$$y^3=a^2x^2.$$

The value of x in this equation is $\frac{y^{\frac{3}{2}}}{a}$.

Differentiating this equation, we have

$$dx=\frac{3y^{\frac{1}{2}}dy}{2a},$$

and consequently,
$$dx^2=\frac{9y}{4a^2}dy^2.$$

Substituting this value in the differential of the arc, we have

$$\sqrt{dx^2+dy^2}=\sqrt{\left(\frac{9y}{4a^2}+1\right)dy^2}=dy\sqrt{\frac{9y+4a^2}{4a^2}}=dy\frac{(9y+4a^2)^{\frac{1}{2}}}{2a}.$$

Integrating by Art. 303, we have

$$\int\sqrt{dx^2+dy^2}=\frac{(9y+4a^2)^{\frac{3}{2}}}{27a}+C.$$

To determine the constant C, we see, from the equation of the curve, that at the origin of abscissas $y=0$, and consequently $z=0$; hence

$$0=\frac{8a^3}{27}+C;$$

whence $$C=-\frac{8a^2}{27},$$

and consequently the entire integral is

$$z=\frac{(9y+4a^2)^{\frac{3}{2}}}{27a}-\frac{8a^2}{27}.$$

Ex. 2. It is required to find the length of an arc of a circle.

We have found (Art. 227) that if z represents an arc of a circle, and t its tangent, we have

$$dz=\frac{dt}{1+t^2}=dt\times\frac{1}{1+t^2}.$$

But $$\frac{1}{1+t^2}=1-t^2+t^4-t^6+, \text{ etc.}$$

Hence $$dz=\frac{dt}{1+t^2}=dt-t^2dt+t^4dt-t^6dt+, \text{ etc.}$$

Hence, integrating each term separately, we obtain

$$\int dz=z=t-\frac{t^3}{3}+\frac{t^5}{5}-\frac{t^7}{7}+\frac{t^9}{9}-, \text{ etc.} =t(1-\frac{t^2}{3}+\frac{t^4}{5}-\frac{t^6}{7}+\frac{t^8}{9}-, \text{ etc.}).$$

If we take z equal to an arc of 30°, its tangent will be $=\sqrt{\frac{1}{3}}$, which equals 0.577350, which being substituted for t in this series, we obtain

$$z=\sqrt{\tfrac{1}{3}}\times(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\frac{1}{9.3^4}-, \text{ etc.})$$
$$=0.5235987,$$

the length of an arc of 30°, which, multiplied by 6, gives the semicircumference to the radius unity

$$=3.141592.$$

Ex. 3. It is required to find the length of an arc of a cycloid.

The differential equation of the cycloid, Art. 275, is

$$dx=\frac{ydy}{\sqrt{2ry-y^2}};$$

whence $$dx^2=\frac{y^2dy^2}{2ry-y^2}.$$

Substituting this value of dx^2 in the differential of the arc, we obtain

$$dz=\sqrt{dy^2+\frac{y^2dy^2}{2ry-y^2}},$$

$$=dy\sqrt{\frac{2ry}{2ry-y^2}}=dy\sqrt{\frac{2r}{2r-y}},$$

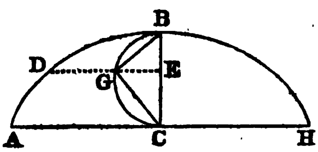
$$=(2r)^{\frac{1}{2}}(2r-y)^{-\frac{1}{2}}dy.$$

Integrating by Art. 303, we obtain

$$\int(2r-y)^{-\frac{1}{2}}dy=-2(2r-y)^{\frac{1}{2}}+C.$$

Hence $$z=-(2r)^{\frac{1}{2}}2\sqrt{2r-y}+C,$$

$$=-2\sqrt{2r(2r-y)}+C.$$



If we estimate the arc from the point B where $y=2r$, we shall have, when $z=0$, $y=2r$; hence

$$0=0+C,$$

or $$C=0,$$

which shows that there is no constant to add, and consequently the entire integral will be

$$z=-2\sqrt{2r(2r-y)},$$

which represents the length of the arc of the cycloid from B to any point D whose co-ordinates are x and y.

But we see from the figure that $BE=2r-y$.

Also, $BG^2=BC\times BE$, Geom., Prop. XXII., B. IV.

Hence $BG=\sqrt{BC\times BE}=\sqrt{2r(2r-y)}$,

and consequently the arc $BD=2BG$,

or *the arc of a cycloid, estimated from the vertex of the axis, is equal to twice the corresponding chord of the generating circle; hence the entire arc* BDA *is equal to twice the diameter* BC, *and the entire curve* ADBH *is equal to four times the diameter of the generating circle.*

Ex. 4. It is required to find the length of an arc of the common parabola.

The equation of the parabola is

$$y^2=2px.$$

Differentiating, and dividing by 2, we have

$$ydy=pdx;$$

whence $$dx^2=\frac{y^2}{p^2}dy^2.$$

Substituting this value in the differential of the arc, we obtain

$$dz=\sqrt{dy^2+\frac{y^2}{p^2}dy^2},$$

$$=\frac{dy}{p}\sqrt{p^2+y^2}.$$

Integrating according to Ex. 1, Art. 322, we obtain

$$z=\frac{y\sqrt{p^2+y^2}}{2p}+\frac{p}{2}\log.(y+\sqrt{p^2+y^2})+C.$$

If we estimate the arc from the vertex of the parabola, we shall have

$$y=0 \text{ when } z=0;$$

hence $$0=\frac{p}{2}\log.p+C, \text{ or } C=-\frac{p}{2}\log.p;$$

and consequently

$$z=\frac{y\sqrt{p^2+y^2}}{2p}+\frac{p}{2}\log.\left(\frac{y+\sqrt{p^2+y^2}}{p}\right).$$

Ex. 5. It is required to find the length of an arc of the logarithmic spiral.

The differential of an arc of a polar curve, referred to polar co-ordinates, Art. 257, is

$$dz=\sqrt{dr^2+r^2dt^2}.$$

The equation of the logarithmic spiral, Art. 155, is

$$t=\log.r.$$

Consequently $$dt=\frac{Mdr}{r}.$$

Hence, by substitution, we find

$$dz=\sqrt{dr^2+M^2dr^2},$$

$$=dr\sqrt{1+M^2}.$$

For the Naperian system $M=1$, and we find

$$dz=dr\sqrt{2};$$

whence $$z=r\sqrt{2}+C.$$

If we estimate the arc from the pole where $r=0$, we have

$$z=r\sqrt{2};$$

that is, in the Naperian logarithmic spiral, *the length of an arc, estimated from the pole to any point of the curve, is equal to the diagonal of a square described on the radius vector.*

Ex. 6. It is required to find the length of an arc of an ellipse.

The equation of an ellipse, Art. 69, Cor. 6, is

$$y^2=(1-e^2)(A^2-x^2).$$

Differentiating, we obtain

$$\frac{dy}{dx}=-\frac{(1-e^2)x}{y},$$

$$=-\frac{x\sqrt{1-e^2}}{\sqrt{A^2-x^2}}.$$

Substituting this value in the differential of the arc, we obtain

$$dz=dx\sqrt{1+\frac{x^2(1-e^2)}{A^2-x^2}},$$

$$=\frac{dx\sqrt{A^2-e^2x^2}}{\sqrt{A^2-x^2}},$$

$$=\frac{Adx\sqrt{1-\frac{e^2x^2}{A^2}}}{\sqrt{A^2-x^2}}.$$

Developing $\sqrt{1-\frac{e^2x^2}{A^2}}$ in a series, we obtain

$$dz=\frac{Adx}{\sqrt{A^2-x^2}}\left(1-\frac{e^2x^2}{2A^2}-\frac{e^4x^4}{2.4A^4}-\frac{3e^6x^6}{2.4.6A^6}-\text{, etc.}\right).$$

The several terms of this series may be integrated as in Art. 317, and we obtain

$$z=X_0-\frac{e^2}{2A}X_2-\frac{e^4}{2.4A^3}X_4-\frac{3e^6}{2.4.6A^5}X_6-\text{, etc.,} \qquad (1)$$

where X_0 represents the arc of a circle whose radius is A and sine is x,

$$X_2=\frac{A.X_0}{2}-\frac{x}{2}\sqrt{A^2-x^2},$$

$$X_4=\frac{3A^2.X_2}{4}-\frac{x^3}{4}\sqrt{A^2-x^2},$$

$$X_6=\frac{5A^2.X_4}{6}-\frac{x^5}{6}\sqrt{A^2-x^2}, \text{ etc.}$$

In order to obtain one fourth of the circumference of the ellipse, we must integrate between the limits $x=0$ and $x=A$. But when $x=A$, $\sqrt{A^2-x^2}=0$; hence the values of the quantities, X_2, X_4, etc., become

$$X_2=\frac{A.X_0}{2},$$

$$X_4=\frac{3A^2.X_2}{4}=\frac{3A^2.X_0}{2.4},$$

$$X_6=\frac{5A^2.X_4}{6}=\frac{3.5A^4.X_0}{2.4.6}, \text{ etc.}$$

and consequently equation (1) becomes

$$z=X_0(1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5e^6}{2.2.4.4.6.6}-, \text{ etc.}),$$

for one fourth of the circumference of the ellipse, where X_0 is one fourth of the circumference of the circle whose radius is A. Hence the entire circumference of the ellipse is equal to

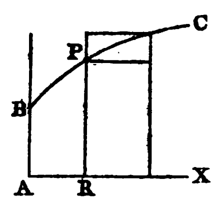
$$2\pi A(1-\frac{e^2}{2.2}-\frac{3e^4}{2.2.4.4}-\frac{3.3.5e^6}{2.2.4.4.6.6}-, \text{ etc.})$$

QUADRATURE OF CURVES.

(325.) The *quadrature* of a curve is the measuring of its area, or the finding a rectilinear space equal to a proposed curvilinear one. When the area of a curve can be expressed in a finite number of algebraic terms, the curve is said to be *quadrable*, and may be represented by an equivalent square.

We have found, Art. 253, that the differential of the area of a segment of any curve, referred to rectangular co-ordinates, is

$$ds=ydx,$$



where s represents the area ABPR, and x and y are the co-ordinates of the point P.

To apply this formula to any particular curve, we must *find from the equation of the curve the value of* y *in terms of* x, *or the value of* dx *in terms of* y *and* dy, *and substitute in the formula* ds=ydx. *The integral of this expression will give the area of the curve.*

Ex. 1. It is required to find the area of the common parabola.

The equation of the parabola is

$$y^2=2px;$$

whence, by differentiating,

$$dx=\frac{ydy}{p}.$$

Therefore $$ydx=\frac{y^2dy}{p},$$

and, by integrating, $$s=\frac{y^3}{3p}+C.$$

If we estimate the area from the vertex of the parabola, the constant C will be equal to zero, because when y is made equal to 0, the surface is equal to 0; hence the entire integral is

$$s=\frac{y^3}{3p},$$

which equals $$\frac{y}{3p}\times y^2=\frac{y}{3p}\times 2px=\tfrac{2}{3}xy;$$

that is, *the area of a segment of a parabola is equal to two thirds of the area of the rectangle described on the abscissa and ordinate.*

Ex. 2. It is required to find the area of any parabola.

The general equation of the parabolas, Art. 136, is

$$y^n=ax;$$

whence, by differentiating, we obtain

$$ny^{n-1}dy=adx,$$

and $$dx=\frac{ny^{n-1}dy}{a}.$$

Therefore $$ydx=\frac{ny^n dy}{a}.$$

And, by integrating, Art. 298,

$$s=\frac{ny^{n+1}}{(n+1)a}+C,$$

or $$s=\frac{n}{n+1}\times\frac{y^n}{a}\times y+C,$$

$$=\frac{n}{n+1}xy+C,$$ by substituting x for its value $\frac{y^n}{a}$.

If we estimate the area from the vertex of the parabola, the constant C will be equal to zero; hence

$$s=\frac{n}{n+1}xy.$$

Hence *the area of any portion of a parabola is equal to the*

rectangle described on the abscissa and ordinate multiplied by the ratio $\frac{n}{n+1}$.

If $n=2$, the equation represents the common parabola, and the area equals $\frac{2}{3}xy$.

If $n=1$, the figure becomes a triangle, and the area equals

$$\tfrac{1}{2}xy;$$

that is, *the area of a triangle is equal to half the product of its base and perpendicular.*

Ex. 3. It is required to find the area of a circle.

The equation of the circle, when the radius equals unity, is

$$y=(1-x^2)^{\frac{1}{2}}.$$

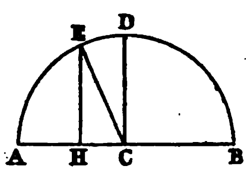
The second member of this equation being developed by the binomial theorem, we have

$$y=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-, \text{ etc.}$$

Hence $$ydx=dx-\frac{x^2dx}{2}-\frac{x^4dx}{8}-\frac{x^6dx}{16}-\frac{5x^8dx}{128}-, \text{ etc.,}$$

and integrating each term separately, we have

$$s=\int ydx=x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-, \text{ etc., } +C.$$



If we estimate the arc from the point D, when $x=0$, the area CDEH is 0, and consequently $C=0$. The preceding series, therefore, expresses the area of the segment CDEH.

If the arc DE be taken equal to 30°, the sine of 30°, or its equal CH, which is x, becomes $=\frac{1}{2}$, and we have

$$\text{CDEH}=\frac{1}{2}-\frac{1}{48}-\frac{1}{1280}-\frac{1}{14336}-, \text{ etc.,}$$
$$=.4783055.$$

But as $x=\frac{1}{2}$, $y=\sqrt{\frac{3}{4}}$; therefore the area of the triangle

$$\text{CEH}=\tfrac{1}{4}\times\sqrt{\tfrac{3}{4}}=.2165063.$$

Hence the area of the sector

$$\text{CDE}=.2617992,$$

which, multiplied by 12, gives

3.14159, etc., for the area of the whole circle.

Ex. 4. It is required to find the area of an ellipse.

The equation of the ellipse, referred to its center and axes, is

$$y=\frac{\mathrm{B}}{\mathrm{A}}\sqrt{\mathrm{A}^2-x^2},$$

and consequently the area of the semi-ellipse will be equal to

$$\int y dx=\frac{\mathrm{B}}{\mathrm{A}}\int dx\sqrt{\mathrm{A}^2-x^2}.$$

But $dx\sqrt{\mathrm{A}^2-x^2}$ is the differential of the area of a circle whose radius is A, Art. 254; hence the area of the ellipse $=\frac{\mathrm{B}}{\mathrm{A}}\times$ the area of the circumscribing circle.

But the area of the circumscribing circle is equal to $\pi\mathrm{A}^2$; hence the area of the ellipse is equal to

$$\pi\mathrm{A}^2\times\frac{\mathrm{B}}{\mathrm{A}},$$

or

$$\pi\mathrm{AB}.$$

Ex. 5. It is required to find the area of a segment of an hyperbola.

The equation of the hyperbola, referred to its center and axes, is

$$\mathrm{A}^2y^2-\mathrm{B}^2x^2=-\mathrm{A}^2\mathrm{B}^2:$$

whence

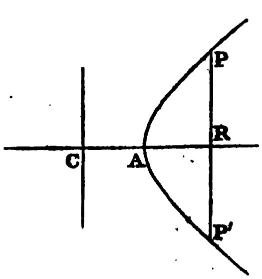
$$y=\frac{\mathrm{B}}{\mathrm{A}}\sqrt{x^2-\mathrm{A}^2}.$$

Consequently

$$ds=ydx=\frac{\mathrm{B}dx}{\mathrm{A}}\sqrt{x^2-\mathrm{A}^2}.$$

Integrating according to Ex. 2, Art. 322, we obtain

$$s=\mathrm{B}x\frac{\sqrt{x^2-\mathrm{A}^2}}{2\mathrm{A}}-\frac{\mathrm{A.B}}{2}\log.[x+\sqrt{x^2-\mathrm{A}^2}]+\mathrm{C}.$$



In order to determine the constant C, make $x=\mathrm{A}$, in which case $s=0$, and we have

$$0=-\frac{\mathrm{A.B}}{2}\log.\mathrm{A}+\mathrm{C};$$

that is,

$$\mathrm{C}=\frac{\mathrm{A.B}}{2}\log.\mathrm{A}.$$

Hence

$$s=\frac{\mathrm{B}x\sqrt{x^2-\mathrm{A}^2}}{2\mathrm{A}}-\frac{\mathrm{A.B}}{2}\log.\left\{\frac{x+\sqrt{x^2-\mathrm{A}^2}}{\mathrm{A}}\right\},$$

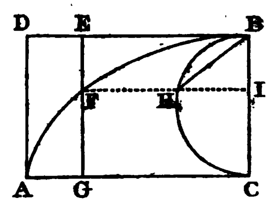
which represents the segment APR; hence the entire segment APP′ is

$$\frac{Bx\sqrt{x^2-A^2}}{A}-A.B\log.\left\{\frac{x+\sqrt{x^2-A^2}}{A}\right\},$$

which equals $$xy-A.B\log.\left\{\frac{x}{A}+\frac{y}{B}\right\},$$

or $$xy-A.B\log.\left\{\frac{Ay+Bx}{A.B}\right\}.$$

Ex. 6. It is required to find the area of a cycloid.



The area of the space ABC is most conveniently obtained by first finding the area of the space ABD, contained between the lines AD, DB, and the convex side of the curve.

Let $BC=2r, AG=x, FG=y$; whence $FE=2r-y=v$. We shall then have

$$d(ADEF)=ds=vdx=(2r-y)dx.$$

But the differential equation of the cycloid, Art. 275, is

$$dx=\frac{ydy}{\sqrt{2ry-y^2}}.$$

Hence $$ds=dy\sqrt{2ry-y^2},$$

and $$s=\int dy\sqrt{2ry-y^2}+C.$$

But this is evidently the area of a segment of a circle whose radius is r and abscissa y (Art. 254); that is, the area of the segment CHI. If we estimate the area of the first segment ADEF from AD, and the area of the segment CHI from the point C, they will both be 0 when $y=0$; the constant C, to be added in each case, will then be 0, and we shall have

$$ADEF=CHI;$$

and when $y=2r$,

$$ADB=\text{the semicircle } CHB=\frac{\pi r^2}{2}.$$

But the area of the rectangle ADBC is equal to

$$AC\times AD=\pi r\times 2r=2\pi r^2.$$

Hence the area $AFBC=ADBC-ADB=\frac{3}{2}\pi r^2=$ three times the semicircle CHB; and doubling this, we find *the area included between one branch of the cycloid and its base, is equal to three times the area of the generating circle.*

AREA OF SPIRALS.

(326.) The differential of the area of a segment of a polar curve, Art. 258, is

$$ds=\frac{r^2dt}{2}.$$

Ex. 1. It is required to find the area of the spiral of Archimedes.

The equation of the spiral of Archimedes is

$$r=\frac{t}{2\pi};$$

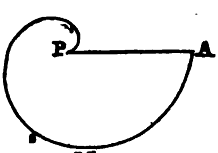
whence $$dr=\frac{dt}{2\pi},$$

$$ds=\pi r^2dr;$$

hence $$s=\frac{\pi r^3}{3}=\frac{t^3}{24\pi^2}.$$

If we make $t=2\pi$, we have

$$s=\frac{\pi}{3},$$

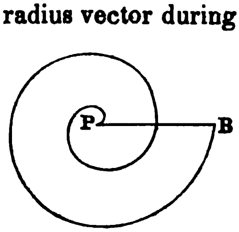


which is the area PMA described by one revolution of the radius vector. Hence the area included by the first spire is equal to one third the area of the circle, whose radius is equal to the radius vector, after the first revolution.

If we make $t=2(2\pi)$, we have

$$s=\frac{8\pi}{3},$$

which is the whole area described by the radius vector during two revolutions. But in the second revolution the radius vector describes the part PMA a second time; hence, to obtain the area PNB, we must subtract that described during the first revolution; hence



$$\text{the area PNB}=\frac{8\pi}{3}-\frac{\pi}{3}=\frac{7\pi}{3}.$$

Ex. 2. It is required to find the area of the hyperbolic spiral.

The equation of the hyperbolic spiral is

$$r=\frac{a}{t};$$

whence $$ds=\frac{a^2dt}{2t^3},$$

and $$s=-\frac{a^2}{2t^2}.$$

Ex. 2. It is required to find the area of the logarithmic spiral.

The equation of the logarithmic spiral is

$$t=\log. r.$$

Hence $$dt=\frac{\mathrm{M}dr}{r}.$$

When M=1, $$ds=\frac{rdr}{2},$$

and $$s=\frac{r^2}{4}+\mathrm{C}.$$

If we estimate the area from the pole where $r=0$ and $\mathrm{C}=0$, we have

$$s=\frac{r^2}{4};$$

that is, *the area, of the Naperian logarithmic spiral is equal to one fourth the square described upon the radius vector.*

AREA OF SURFACES OF REVOLUTION.

(327.) We have found (Art. 255) that the differential of the area of a surface of revolution is

$$d\mathrm{S}=2\pi y\sqrt{dx^2+dx^2};$$

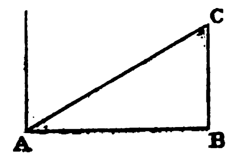
whence $$\mathrm{S}=\int 2\pi y\sqrt{dx^2+dy^2}, \qquad (1)$$

which is a general expression for the area of an indefinite portion of a surface of revolution; the axis of X being the axis of revolution, and $\sqrt{dx^2+dy^2}$ the differential of the arc of the generating curve.

In order to obtain the area of any particular surface, we differentiate the equation of the generating curve, and deduce from it the values of y and dy in terms of x and dx; or of dx in terms of y and dy, which we substitute in expression (1). The integral of this expression will be the area required.

Ex. 1. It is required to determine the convex surface of a cone.

If the right-angled triangle ABC be revolved about AB, the hypothenuse AC will describe the convex surface of a cone. Let $AB=h$, $BC=b$, and let x and y be the co-ordinates of any point of the line AC, referred to the point A as an origin; we shall then have



$$x : y :: h : b;$$

whence
$$y=\frac{bx}{h}.$$

By differentiation, we obtain

$$dy=\frac{b}{h}dx, \text{ and } dy^2=\frac{b^2}{h^2}dx^2.$$

Substituting these values of y and dy^2 in the general formula, we have

$$\int 2\pi y\sqrt{dx^2+dy^2}=\int 2\pi\frac{bx}{h^2}dx\sqrt{h^2+b^2},$$
$$=\pi\frac{bx^2}{h^2}\sqrt{h^2+b^2}+C.$$

If we estimate the area from the vertex where $x=0$, we have $C=0$, and

$$S=\pi\frac{bx^2}{h^2}\sqrt{h^2+b^2}.$$

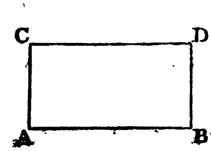
Making $x=AB=h$, we have the surface of the cone whose altitude is h, and the radius of its base b,

$$\pi b\sqrt{h^2+b^2}=2\pi b\times\frac{AC}{2};$$

that is, *the convex surface of a cone is equal to the circumference of its base into half its side.*

Ex. 2. It is required to determine the convex surface of a cylinder.

If the rectangle ABCD be revolved about the side AB, the side CD will describe the convex surface of a cylinder. Let $AB=h$, and $CA=b$; the equation of the straight line CD will be $y=b$; whence



$$dy=0.$$

Substituting these values in the general formula, we obtain

$$\int 2\pi y\sqrt{dx^2+dy^2}=\int 2\pi b dx,$$
$$=2\pi bx+C.$$

If we estimate the area from the point A where $x=0$, C becomes equal to 0; and if we make $x=AB=h$, we have the convex surface of the cylinder

$$2\pi bh;$$

that is, *the convex surface of a cylinder is equal to the circumference of its base into its altitude.*

Ex. 3. It is required to determine the surface of a sphere.

The equation of the generating circle, referred to the center as an origin, is

$$x^2+y^2=R^2.$$

By differentiating, we obtain

$$xdx+ydy=0;$$

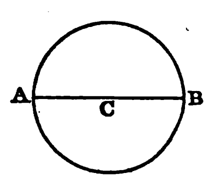
whence
$$dy=-\frac{xdx}{y},$$

and
$$dy^2=\frac{x^2dx^2}{y^2}.$$

Substituting this value in the general formula, we obtain

$$\int 2\pi y\sqrt{\left(\frac{x^2}{y^2}+1\right)dx^2}=\int 2\pi dx\sqrt{x^2+y^2},$$
$$=\int 2\pi R dx,$$
$$=2\pi Rx+C. \qquad (1)$$

To determine the constant, we will suppose the integral to commence at the center of the sphere; and since the origin of co-ordinates is at the center, the integral will be zero when $x=0$, and therefore the constant is equal to zero. Making $x=R$, we have for the surface of a hemisphere



$$2\pi R^2,$$

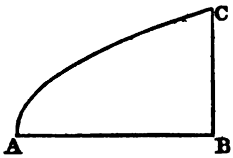
and therefore the surface of the sphere is

$$4\pi R^2;$$

that is, *the surface of a sphere is equal to four of its great circles.*

Ex. 4. It is required to determine the surface of a paraboloid.

A paraboloid is a solid described by the revolution of an arc AC of a parabola about its axis AB.



The equation of the parabola is

$$y^2=2px,$$

which, being differentiated, gives

$$dx=\frac{ydy}{p}, \text{ and } dx^2=\frac{y^2dy^2}{p^2}.$$

Substituting this value in the general formula, it reduces to

$$dS=2\pi y\sqrt{\left(\frac{y^2+p^2}{p^2}\right)dy^2}=\frac{2\pi}{p}ydy\sqrt{y^2+p^2}.$$

Integrating according to Art. 303, we obtain

$$S=\frac{2\pi}{3p}(y^2+p^2)^{\frac{3}{2}}+C.$$

To determine the constant C, let us suppose that y becomes zero, in which case S also reduces to zero, and the preceding equation becomes

$$0=\frac{2\pi p^2}{3}+C;$$

whence $$C=-\frac{2\pi p^2}{3};$$

and supposing the integral to be taken between the limits $y=0$ and $y=b$, the entire integral will be

$$\frac{2\pi}{3p}[(b^2+p^2)^{\frac{3}{2}}-p^3].$$

Ex. 5. It is required to determine the surface of an ellipsoid, described by revolving an ellipse about its major axis.

According to Art. 255, we have

$$dS=2\pi y\sqrt{dx^2+dy^2},$$

or $$dS=2\pi ydz.$$

But in Ex. 6, Art. 324, we have found

$$dz=\frac{Adx}{\sqrt{A^2-x^2}}\left(1-\frac{e^2x^2}{2A^2}-\frac{e^4x^4}{2.4A^4}-\frac{3e^6x^6}{2.4.6A^6}-, \text{etc.}\right);$$

hence $$dS=\frac{2\pi Aydx}{\sqrt{A^2-x^2}}\left(1-\frac{e^2x^2}{2A^2}-\frac{e^4x^4}{2.4A^4}-\frac{3e^6x^6}{2.4.6A^6}-, \text{etc.}\right).$$

But $$\frac{Ay}{\sqrt{A^2-x^2}}=B.$$

Hence $$dS=2\pi Bdx\left(1-\frac{e^2x^2}{2A^2}-\frac{e^4x^4}{2.4A^4}-\frac{3e^6x^6}{2.4.6A^6}-, \text{etc.}\right),$$

and integrating each term separately, we obtain

$$S=2\pi Bx\left(1-\frac{e^2x^2}{2.3A^2}-\frac{e^4x^4}{2.4.5A^4}-\frac{3e^6x^6}{2.4.6.7A^6}-\text{, etc.}\right)+C.$$

Integrating between the limits $x=0$ and $x=A$, we shall obtain half the surface of the ellipsoid

$$=2\pi AB\left(1-\frac{e^2}{2.3}-\frac{e^4}{2.4.5}-\frac{3e^6}{2.4.6.7}-\text{, etc.}\right),$$

or the entire surface of the ellipsoid equals

$$4\pi AB\left(1-\frac{e^2}{2.3}-\frac{e^4}{2.4.5}-\frac{3e^6}{2.4.6.7}-\text{, etc.}\right).$$

Ex. 6. It is required to determine the surface described by the revolution of a cycloid about its base.

The general formula for the differential of the surface is

$$dS=2\pi y dz.$$

But we have found in Ex. 3, Art. 324.

$$dz=dy\sqrt{\frac{2ry}{2ry-y^2}}.$$

Hence $$dS=2\pi y dy\sqrt{\frac{2ry}{2ry-y^2}}=\frac{2\pi\sqrt{2r}\,y^{\frac{3}{2}}dy}{\sqrt{2ry-y^2}},$$

which, being integrated, will give the value of the surface required.

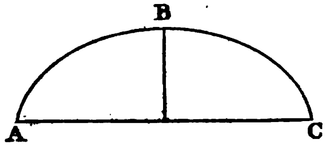
But, according to Ex. 6, Art. 320,

$$\int\frac{y^{\frac{3}{2}}dy}{\sqrt{2ry-y^2}}=-\frac{8r}{3}\sqrt{2r-y}-\frac{2y}{3}\sqrt{2r-y}.$$

Hence

$$\int\frac{2\pi\sqrt{2r}\,y^{\frac{3}{2}}dy}{\sqrt{2ry-y^2}}=2\pi\sqrt{2r}\left[-\frac{8r}{3}\sqrt{2r-y}-\frac{2y}{3}\sqrt{2r-y}\right]+C.$$

If we estimate the surface from the plane passing through B, we shall have $S=0$ when $y=2r$, and consequently $C=0$. If we then integrate between the limits



$$y=0 \text{ and } y=2r,$$

we have half the surface $=\frac{32}{3}\pi r^2$;

hence the entire surface $=\frac{64}{3}\pi r^2$;

that is, *the surface described by the cycloid revolved about its base, is equal to 64 thirds of the generating circle.*

CUBATURE OF SOLIDS OF REVOLUTION.

(328.) The *cubature* of a solid is the finding its solid contents, or finding a cube to which it is equal.

We have found, Art. 256, that the differential of a solid of revolution is

$$dV=\pi y^2dx;$$

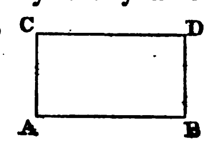
whence $$V=\int\pi y^2dx, \qquad (1)$$

where x and y represent the co-ordinates of the curve which generates the bounding surface, the axis of X being the axis of revolution.

For the cubature of any particular solid, we differentiate the equation of the generating curve, and deduce from it the value of dx in terms of y and dy, or the value of y^2 in terms of x which we substitute in expression (1). The integral of this expression will be the solid required.

Ex. 1. It is required to determine the solidity of a cylinder.

Let r represent AC, the radius of the base, and h the altitude AB. Then



$$V=\int\pi y^2dx=\int\pi b^2dx,$$
$$=\pi b^2x+C.$$

Taking the integral between the limits $x=0$ and $x=AB=h$, we have

$$V=\pi b^2h;$$

that is, *the solidity of a cylinder is equal to the product of its base by its altitude.*

Ex. 2. It is required to determine the solidity of a cone.

Let h represent the altitude of the cone, and r the radius of its base. We shall then have

$$y=\frac{r}{h}x, \text{ and } y^2=\frac{r^2}{h^2}x^2.$$

Substituting this value of y^2 in the general formula, it becomes

$$dV=\frac{r^2}{h^2}\pi x^2dx;$$

whence $$V=\frac{r^2\pi x^3}{3h^2}+C.$$

And taking the integral between the limits $x=0$ and $x=h$, we obtain

$$V=\tfrac{1}{3}\pi r^2 h=\pi r^2\times\frac{h}{3};$$

that is, *the solidity of a cone is equal to the area of its base into one third of its altitude.*

Ex. 3. It is required to find the solidity of a prolate spheroid, or the solid described by the revolution of an ellipse about its major axis.

The equation of an ellipse is

$$y^2=\frac{B^2}{A^2}(A^2-x^2).$$

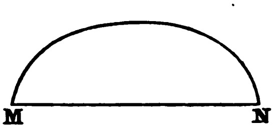
Substituting this value of y^2 in the general formula, it becomes

$$dV=\pi\frac{B^2}{A^2}(A^2-x^2)dx,$$

and by integrating, we find

$$V=\pi\frac{B^2}{A^2}\left(A^2x-\frac{x^3}{3}\right)+C.$$

If we estimate the solidity from the plane passing through the center perpendicular to the major axis, we shall have when $x=0$, $V=0$, and consequently $C=0$. Therefore



$$V=\frac{\pi B^2}{A^2}\left(A^2x-\frac{x^3}{3}\right).$$

Making $x=A$, we obtain for one half of the spheroid

$$\tfrac{2}{3}\pi B^2A;$$

and consequently the entire spheroid equals

$$\tfrac{4}{3}\pi B^2A,\text{ or }\tfrac{2}{3}\pi B^2\times 2A.$$

But πB^2 represents the area of a circle described upon the minor axis, and $2A$ is the major axis; hence *the solidity of a prolate spheroid is equal to two thirds of the circumscribing cylinder.*

Cor. If we make $A=B$, we obtain the solidity of the sphere,

$$\tfrac{4}{3}\pi R^3=\tfrac{1}{6}\pi D^3.$$

Ex. 4. It is required to find the solidity of the common paraboloid.

The equation of the parabola is

$$y^2=2px.$$

Substituting this value of y^2 in the general formula, it becomes

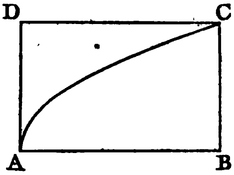
$$dV=2\pi pxdx.$$

Hence $$V=\pi px^2+C.$$

To determine the constant, we suppose x to become equal to zero, in which case the solidity is zero, and $C=0$.

Taking the integral between the limits $x=0$ and $x=h$, and designating by b, the ordinate corresponding to the abscissa $x=h$, we have

$$V=\pi ph^2=\pi b^2\times\frac{h}{2}.$$



But πb^2 represents the area of a circle of which BC is the radius; hence *the solidity of the paraboloid is one half that of the circumscribed cylinder.*

Ex. 5. It is required to find the solidity of the solid generated by the revolution of any parabola about its axis.

The general equation of the parabolas is

$$y^n=ax;$$

whence $$dx=\frac{ny^{n-1}dy}{a},$$

and $$dV=\frac{\pi ny^{n+1}dy}{a}.$$

Hence $$V=\frac{\pi ny^{n+2}}{(n+2)a}+C,$$

$$=\frac{n}{n+2}\times\pi y^2\times\frac{y^n}{a}+C,$$

$$=\frac{n}{n+2}\pi y^2x+C.$$

But when $x=0$, $V=0$, and therefore $C=0$; hence

$$V=\frac{n}{n+2}\pi y^2x.$$

If $n=2$, the solid becomes the common paraboloid, and its solidity equals $\frac{1}{2}\pi y^2x$.

If $n=1$, the curve becomes a straight line, and the solid becomes a cone, and its solidity equals $\frac{1}{3}\pi y^2x$.

Ex. 6. It is required to find the solidity of the solid generated by the revolution of the cycloid about its base.

The general formula for the differential of a solid of revolution is

$$dV=\pi y^2 dx.$$

But we have found for the cycloid, Art. 275,

$$dx=\frac{y\,dy}{\sqrt{2ry-y^2}}.$$

Hence
$$dV=\frac{\pi y^3 dy}{\sqrt{2ry-y^2}},$$

which being integrated, will give the value of the solid required.

The integral of this expression has already been found in Art. 3[illegible].

Hence
$$V=\pi\left(\frac{5r}{3}X_2-\frac{y^2}{3}\sqrt{2ry-y^2}\right),$$

where
$$X_2=\frac{3r}{2}X_1-\frac{y}{2}\sqrt{2ry-y^2},$$

$$X_1=X_0-\sqrt{2ry-y^2},$$

$X_0=$ arc of which r is the radius and y the versed sine.

We must now integrate between the limits $y=0$ and $y=2r$.

When $y=0$, all the above terms become 0.

When $y=2r$, these values become

$$X_0=\pi r,$$

$$X_1=X_0=\pi r,$$

$$X_2=\frac{3r}{2}X_1=\frac{3\pi r^2}{2},$$

and
$$V=\frac{5\pi^2 r^3}{2},$$

which is one half of the solidity; hence $5\pi^2 r^3$ is the solid required.

But $\pi(2r)^2$ represents the base of the circumscribing cylinder,

$2\pi r$ represents its altitude,

and $8\pi^2 r^3$ represents its solidity.

Hence *the solid required is equal to five eighths of the circumscribing cylinder.*

MISCELLANEOUS EXAMPLES.

(329.) Ex. 1. In a right-angled triangle, having given the hypothenuse (a), and the difference between the base and perpendicular ($2d$), to determine the two sides.

$$Ans.\ \sqrt{\frac{a^2-2d^2}{2}}+d, \text{ and } \sqrt{\frac{a^2-2d^2}{2}}-d.$$

Ex. 2. Having given the area (c) of a rectangle inscribed in a triangle whose base is (b) and altitude (a), to determine the height of the rectangle.

$$Ans.\ \frac{a}{2}\pm\sqrt{\frac{a^2}{4}-\frac{ac}{b}}.$$

Ex. 3. Having given the ratio of the two sides of a triangle, as m to n, together with the segments of the base a and b, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

$$Ans.\ m\sqrt{\frac{a^2-b^2}{m^2-n^2}}, \text{ and } n\sqrt{\frac{a^2-b^2}{m^2-n^2}}.$$

Ex. 4. Having given the base of a triangle ($2a$), the sum of the other two sides ($2s$), and the line (c) drawn from the vertical angle to the middle of the base, to find the sides of the triangle.

$$Ans.\ s\pm\sqrt{a^2+c^2-s^2}.$$

Ex. 5. Having given the two sides (a) and (b) about the vertical angle of a triangle, together with the line (c) bisecting that angle and terminating in the base, to find the segments of the base.

$$Ans.\ a\sqrt{\frac{ab-c^2}{ab}}, \text{ and } b\sqrt{\frac{ab-c^2}{ab}}.$$

Ex. 6. Determine the sides of a right-angled triangle, having given its perimeter ($2p$), and the radius (r) of the inscribed circle.

Ans. The hypothenuse equals $p-r$, and the other sides are $\frac{p+r\pm\sqrt{(p-r)^2-4pr}}{2}$.

Ex. 7. Determine the radii of three equal circles, described in a given circle, which touch each other, and also the circumference of the given circle whose radius is R.

Ans. $R(2\sqrt{3}-3)$.

Ex. 8. Having given the three lines a, b, and c, drawn from the three angles of a triangle to the middle of the opposite sides, to determine the sides.

$$\text{Ans. } \tfrac{2}{3}\sqrt{2a^2+2b^2-c^2},$$
$$\tfrac{2}{3}\sqrt{2a^2+2c^2-b^2},$$
$$\tfrac{2}{3}\sqrt{2b^2+2c^2-a^2}.$$

Ex. 9. Having given the hypothenuse (a) of a right-angled triangle, and the radius (r) of the inscribed circle, to determine the other sides.

$$\text{Ans. } \tfrac{1}{2}(a+2r\pm\sqrt{a^2-4ar-4r^2}).$$

Ex. 10. Draw the line whose equation is

$$y^2=5x^2.$$

Ex. 11. Construct the equation

$$y^2-7y+12=0.$$

Ex. 12. Describe the circle whose equation is

$$y^2+x^2+4y-4x=8.$$

Ex. 13. Describe the circle whose equation is

$$y^2+x^2-6y+8x=11.$$

Ex. 14. Prove that the three perpendiculars drawn from the angles to the opposite sides of a triangle, pass through the same point.

Ex. 15. Prove that the three straight lines drawn from the angles of a triangle to bisect the opposite sides, pass through the same point.

Ex. 16. What is the differential of the function

$$u=(a+bx^2)^3x^5\,?$$

$$\text{Ans. } du=5(a+bx^2)^3x^4dx+6b(a+bx^2)^2x^6dx.$$

Ex. 17. What is the differential of the function

$$u=x(a+x)(a^2+x^2)\,?$$

$$\text{Ans. } du=(a^3+2a^2x+3ax^2+4x^3)dx.$$

Ex. 18. What is the differential of the function

$$u=x(a^2+x^2)\sqrt{a^2-x^2}\,?$$

$$Ans.\ du=\frac{(a^4+a^2x^2-4x^4)dx}{\sqrt{a^2-x^2}}.$$

Ex. 19. What is the differential of the function

$$u=(a+\sqrt{x})^3\,?$$

$$Ans.\ du=\frac{3(a+\sqrt{x})^2dx}{2\sqrt{x}}.$$

Ex. 20. What is the differential of the function

$$u=\frac{a+x}{\sqrt{a-x}}\,?$$

$$Ans.\ du=\frac{3a-x}{2(a-x)^{\frac{3}{2}}}.dx.$$

Ex. 21. What is the differential of the function

$$u=\frac{x}{x+\sqrt{1-x^2}}\,?$$

$$Ans.\ du=\frac{dx}{\sqrt{1-x^2}(1+2x\sqrt{1-x^2})}.$$

Ex. 22. What is the differential of the function

$$u=a+\frac{4\sqrt{x}}{3+x^2}\,?$$

$$Ans.\ du=\frac{6(1-x^2)dx}{(3+x^2)^2\sqrt{x}}.$$

Ex. 23. What is the differential of the function

$$u=\frac{a^2-x^2}{a^4+a^2x^2+x^4}\,?$$

$$Ans.\ du=\frac{-2x(2a^4+2a^2x^2-x^4)dx}{(a^4+a^2x^2+x^4)^2}.$$

Ex. 24. What is the differential of the function

$$u=(a+bx)^2(m+nx)^3\,?$$

$$Ans.\ du=3n(a+bx)^2(m+nx)^2dx+2b(a+bx)(m+nx)^3dx.$$

Ex. 25. What is the differential of the function

$$u=\frac{x^n}{(1+x)^n}\,?$$

$$Ans.\ \frac{nx^{n-1}dx}{(1+x)^{n+1}}.$$

Ex. 26. What is the differential of the function

$$u=(a-x)\sqrt{a^2+x^2}?$$

Ans. $du=-\dfrac{(a^2-ax+2x^2)dx}{\sqrt{a^2+x^2}}.$

Ex. 27. What is the differential of the function

$$u=(a^2-x^2)\sqrt{a+x}?$$

Ans. $du=\frac{1}{2}(a-5x)\sqrt{a+x}.dx.$

Ex. 28. What is the differential of the function

$$u=(a+bx^2)^3(c+ex^4)^5?$$

Ans. $du=20(a+bx^2)^3(c+ex^4)^4ex^3dx+6b(c+ex^4)^5(a+bx^2)^2xdx.$

Ex. 29. What is the differential of the function

$$u=\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}?$$

Ans. $du=-\dfrac{a^2+a\sqrt{a^2-x^2}}{x^2\sqrt{a^2-x^2}}dx.$

Ex. 30. What is the differential of the function

$$u=\frac{ay}{\sqrt{x^2+y^2}}?$$

Ans. $du=\dfrac{axydx-ax^2dy}{(x^2+y^2)^{\frac{3}{2}}}.$

Ex. 31. What is the differential of the function

$$u=(2a^2+3x^2)(a^2-x^2)^{\frac{3}{2}}?$$

Ans. $du=-15x^3\sqrt{a^2-x^2}.dx.$

Ex. 32. What is the differential of the function

$$u=x\log. x?$$

Ans. $du=(1+\log. x)dx.$

Ex. 33. What is the differential of the function

$$u=\log. [x+\sqrt{x^2+a^2}]?$$

Ans. $du=\dfrac{dx}{\sqrt{x^2+a^2}}.$

Ex. 34. What is the differential of the function

$$u=\log. \left\{\frac{x}{a+\sqrt{a^2+x^2}}\right\}?$$

Ans. $du=\dfrac{adx}{x\sqrt{a^2+x^2}}.$

Ex. 35. What is the differential of the function

$$u=\frac{\log. x}{x}?$$

Ans. $du=\frac{(1-\log. x)dx}{x^2}$.

Ex. 36. What is the differential of the function

$$u=\log. \left\{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right\}^{\frac{1}{2}}?$$

Ans. $du=\frac{dx}{\sqrt{1+x^2}}$.

Ex. 37. What is the differential of the function

$$u=\frac{x}{\log. x}?$$

Ans. $du=\frac{(\log. x-1)dx}{(\log. x)^2}$.

Ex. 38. What is the differential of the function

$$u=(\log. x)^n?$$

Ans. $du=\frac{n(\log. x)^{n-1}dx}{x}$.

Ex. 39. What is the differential of the function

$$u=\log. \left\{\frac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}\right\}?$$

Ans. $du=-\frac{adx}{x\sqrt{a^2-x^2}}$.

Ex. 40. What is the differential of the function

$$u=\log. \left\{\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}\right\}?$$

Ans. $du=\frac{\sqrt{a}.dx}{(a-x)\sqrt{x}}$.

Ex. 41. What is the differential of the function

$$u=\frac{a+2bx}{(a+bx)^2}?$$

Ans. $du=\frac{-2b^2xdx}{(a+bx)^3}$.

Ex. 42. Develop into a series the function

$$u=\sqrt{a^2+x^2}.$$

Ans. $u=a+\frac{x^2}{2a}-\frac{x^4}{2.4a^3}+\frac{3x^6}{2.4.6a^5}-$, etc.

Ex. 43. Develop into a series the function

$$u=\sqrt{2x-1}.$$

Ans. $u=\sqrt{-1}\{1-x-\frac{x^2}{2}-\frac{x^3}{4}-$, etc.$\}$.

Ex. 44. Develop into a series the function

$$u=\frac{1}{\sqrt{a^4+x^4}}.$$

Ans. $u=\frac{1}{a}-\frac{x^4}{4a^5}+\frac{5x^8}{4.8a^9}-\frac{5.9x^{12}}{4.8.12a^{13}}+\frac{5.9.13x^{16}}{4.8.12.16.a^{17}}-$, etc.

Ex. 45. Determine the maxima and minima values of the function

$$u=\frac{a^2x}{(a-x)^2}.$$

Ans. u has a maximum when $x=+a$, and a minimum when $x=-a$.

Ex. 46. Find the values of x which will render the function

$$u=3a^2x^3-b^4x+c^5$$

a maximum or a minimum.

Ans. There is a maximum corresponding to $x=-\frac{b^2}{3a}$, and a minimum corresponding to $x=+\frac{b^2}{3a}$.

Ex. 47. Required the least parabola which shall circumscribe a circle whose radius is R.

Ans. The axis of the parabola is $\frac{9}{4}$R, and its base is 3R.

Ex. 48. It is required to inscribe the greatest rectangle in an ellipse whose axes are 2A and 2B.

Ans. The sides of the rectangle are $A\sqrt{2}$ and $B\sqrt{2}$.

Ex. 49. It is required to find the fraction that exceeds its cube by the greatest possible quantity.

Ans. $+\sqrt{\frac{1}{3}}$.

Ex. 50. The equation of a certain curve is

$$a^2y=ax^2-x^3;$$

required its greatest and least ordinates.

Ans. When $x=\frac{2}{3}a$, y is a maximum; when $x=0$, y is a minimum.

Ex. 51. What is the altitude of the maximum cylinder which can be inscribed in a given paraboloid?

Ans. Half the axis of the paraboloid.

Ex. 52. Find the values of x which will render u a maximum or a minimum in the equation

$$u=3x^4-16x^3+6x^2+72x-1.$$

Ans. This function has a maximum value when $x=+2$, and a minimum value when $x=-1$ or $+3$.

Ex. 53. It is required to circumscribe about a given parabola, an isosceles triangle whose area shall be a minimum.

Ans. The altitude of the triangle is four thirds of the axis of the parabola.

Ex. 54. Required the subtangent of the curve whose equation is

$$y^2=\frac{x^3}{a-x}.$$

Ans. $\frac{2x(a-x)}{3a-2x}$.

Ex. 55. Required the subtangent of the curve whose equation is

$$xy^2=a^2(a-x).$$

Ans. $-\frac{2(ax-x^2)}{a}$.

Ex. 56. Determine when the subtangent of the preceding curve is a minimum.

Ans. When $x=\frac{1}{2}a$.

Ex. 57. Find the value of the subnormal of the curve whose equation is

$$3ay^2+a^3=2x^3.$$

Ans. $\frac{x^2}{a}$.

Ex. 58. Find the value of the subnormal of the curve whose equation is

$$y^2=2a^2 \log. x.$$

Ans. $\frac{a^2}{x}$.

Ex. 59. Determine the radius of curvature at any point of the cubical parabola whose equation is

$$y^3=ax.$$

Ans. $R=\frac{(9y^4+a^2)^{\frac{3}{2}}}{6a^2y}$.

Ex. 60. Determine when the curvature of the preceding curve is greatest.

Ans. When $y=\sqrt[4]{\frac{a^2}{45}}$.

Ex. 61. Determine the radius of curvature at any point of the logarithmic curve whose equation is

$$y=a^x.$$

Ans. $R=\frac{(M^2+y^2)^{\frac{3}{2}}}{My}$, M being the modulus.

Ex. 62. Determine the point of greatest curvature of the logarithmic curve.

Ans. The point whose ordinate is equal to $\frac{M}{\sqrt{2}}$.

Ex. 63. Determine whether the curve whose equation is

$$(y-b)^3=(x-a)^2,$$

has a cusp at the point where the tangent is parallel to the axis of Y.

Ex. 64. Determine the point of inflection in the curve whose equation is

$$ax^2=a^2y+x^2y.$$

Ans. There is an inflection at the point where $y=\frac{1}{4}a$.

Ex. 65. Determine whether the curve whose equation is

$$y^5=x^3,$$

has a point of inflection.

Ans. This curve has a point of inflection at the origin.

Ex. 66. Required the equation of the curve whose area is equal to twice the rectangle of its co-ordinates.

Ans. The equation is $xy^2=a$.

Ex. 67. Find the integral of the differential

$$du=\frac{dx}{(a-x)^5}.$$

$$Ans.\ u=\frac{1}{4(a-x)^4}.$$

Ex. 68. Find the integral of the differential

$$du=\frac{4xdx}{(1-x^2)^2}.$$

$$Ans.\ u=\frac{2}{1-x^2}+C.$$

Ex. 69. Find the integral of the differential

$$du=\frac{2adx}{x\sqrt{2ax-x^2}}.$$

$$Ans.\ u=-\frac{2\sqrt{2ax-x^2}}{x}+C.$$

Ex. 70. Find the integral of the differential

$$du=\frac{xdx}{(2ax-x^2)^{\frac{3}{2}}}.$$

$$Ans.\ u=\frac{1}{a}\sqrt{\frac{x}{2a-x}}+C.$$

Ex. 71. Find the integral of the differential

$$du=\frac{x^8dx}{\sqrt{a^9+6x^9}}.$$

$$Ans.\ u=\frac{\sqrt{a^9+6x^9}}{27}+C.$$

Ex. 72. Find the integral of the differential

$$du=\frac{dx}{\sqrt{1+x^2}}.$$

$$Ans.\ u=x-\frac{x^3}{2.3}+\frac{3x^5}{3.4.5}-\frac{3.5x^7}{2.4.6.7}+\text{ etc.},+C.$$

Ex. 73. Find the integral of the differential

$$du=\frac{3x^2dx}{x^3+a^3}.$$

Ans. $u=\log.\ (x^3+a^3)$.

Ex. 74. Find the integral of the differential

$$du=\frac{5x^3dx}{3x^4+7}.$$

Ans. $u=\frac{5}{12}\log.\ (3x^4+7)$.

Ex. 75. Find the integral of the differential

$$du=x^3(a+bx^2)^{\frac{1}{2}}dx.$$

Ans. $u=\left(\frac{a+bx^2}{5}-\frac{a}{3}\right)\frac{a+bx^2}{b^2}.$

Ex. 76. Find the integral of the differential

$$du=x^{-2}(a+x^3)^{-\frac{5}{3}}dx.$$

Ans. $u=-\frac{3x^3+2a}{2a^2x(a+x^3)^{\frac{2}{3}}}.$

Ex. 77. Find the integral of the differential

$$du=x^3(a^2+x^2)^{\frac{1}{3}}dx.$$

Ans. $u=\frac{3}{56}(a^2+x^2)^{\frac{4}{3}}(4x^2-3a^2)$.

Ex. 78. Find the integral of the differential

$$du=\frac{adx}{(1+x^2)^{\frac{3}{2}}}.$$

Ans. $u=\frac{ax}{\sqrt{1+x^2}}.$

Ex. 79. Find the integral of the differential

$$du=\frac{x^6dx}{\sqrt{a^2+x^2}}.$$

Ans. $u=\frac{x^5}{6}\sqrt{a^2+x^2}-\frac{5a^2}{6}\int\frac{x^4dx}{\sqrt{a^2+x^2}}.$

Ex. 80. Find the integral of the differential

$$du=\frac{x^{10}dx}{\sqrt{a^2-x^2}}.$$

Ans. $u=\frac{9a^2}{10}\int\frac{x^8dx}{\sqrt{a^2-x^2}}-\frac{x^9}{10}\sqrt{a^2-x^2}.$

Ex. 81 Find the integral of the differential

$$du=\frac{x^5dx}{\sqrt{2ax-x^2}}.$$

$$Ans.\ u=\frac{9a}{5}\int\frac{x^4dx}{\sqrt{2ax-x^2}}-\frac{x^4}{5}\sqrt{2ax-x^2}.$$

Ex. 82. Find the integral of the differential

$$du=\frac{x^5dx}{x^2+a^2}.$$

$$Ans.\ u=\frac{x^4}{4}-\frac{a^2x^2}{2}+\frac{a^4}{2}\log.\ (x^2+a^2).$$

Ex. 83. Find the integral of the differential

$$du=\frac{x^3dx}{\sqrt{1-x^2}}.$$

$$Ans.\ u=\tfrac{2}{3}\int\frac{xdx}{\sqrt{1-x^2}}-\frac{x^2}{3}\sqrt{1-x^2}.$$

Ex. 84. Find the integral of the differential

$$du=\frac{x^3dx}{\sqrt{a+bx^2}}.$$

$$Ans.\ u=\left\{\frac{x^2}{3b}-\frac{2a}{3b^2}\right\}\sqrt{a+bx^2}.$$

Ex. 85. Find the integral of the differential

$$du=x^4(a+bx^3)^{\frac{1}{2}}dx.$$

$$Ans.\ u=\frac{2x^2(a+bx^3)^{\frac{3}{2}}}{13b}-\frac{4a}{13b}\int x(a+bx^3)^{\frac{1}{2}}dx$$

Ex. 86. Find the integral of the differential

$$du=\frac{3x^2+2x+1}{x^3+x^2+x+1}dx.$$

$$Ans.\ u=\log.\ (x^3+x^2+x+1).$$

Ex. 87. Determine the volume of a parabolic spindle which is generated by the revolution of a parabola about its base b, the height being h.

$$Ans.\ V=\frac{16\pi bh^2}{15}.$$

Ex. 88. Determine the area of the logarithmic curve.

$$Ans.\ s=M(y-a).$$

Ex. 89. Determine the volume of the solid generated by the revolution of the logarithmic curve about the axis of X.

Ans. $V=\frac{M\pi y^2}{2}$.

Ex. 90. Determine the volume of the solid generated by the revolution of the curve whose equation is

$$y^2=\frac{x^3}{a-x}.$$

Ans. $V=\pi a^3 \log. \frac{a}{a-x}-\frac{\pi x^3}{3}-\frac{\pi a x^2}{2}-\pi a^2 x.$

THE END.

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